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# Relativistic resonances as non-orthogonal states in Hilbert space

W. Blum[a], H. Saller[b]

Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, München, Germany

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**Abstract.** We analyze the energy-momentum properties of relativistic short-lived particles with the result that they are characterized by two 4-vectors: in addition to the familiar energy-momentum vector (timelike) there is an energy-momentum 'spread vector' (spacelike). The wave functions in space and time for unstable particles are constructed. For the relativistic properties of unstable states we refer to Wigner's method of Poincaré group representations that are induced by representations of the space-time translation and rotation groups. If stable particles, unstable particles and resonances are treated as elementary objects that are not fundamentally different one has to take into account that they will not generally be orthogonal to each other in their state space. The scalar product between a stable and an unstable state with otherwise identical properties is calculated in a particular Lorentz frame. The spin of an unstable particle is not infinitely sharp but has a 'spin spread' giving rise to 'spin neighbors'. This opens the possibility of a non-zero scalar product between states with unequal spin. – A first practical application of non-orthogonal states is seen in diffraction dissociation reactions whose large cross-sections are attributed to interference of states that are 'partially identical'.

## 1 Introduction

From an experimental point of view, stable and unstable particles are often treated alike; for example, in multi-particle reactions one measures the cross section for the production of a $\rho$-meson or an $N^*$-resonance or a pion or a proton without regard to their lifetimes. There is no fundamental difference between stable and unstable states – this is our perspective in the following article[1]. From an algebraic point of view, particle states are eigenvectors of a time translation operator (Hamiltonian). Hermitian Hamiltonians acting on a Hilbert space have real energy eigenvalues, and the eigenstates are orthogonal to each other. If the particle is unstable and therefore the eigenvalue $E - i\Gamma/2$ is complex ($\Gamma > 0$), the Hamiltonian can no longer be Hermitian, and the eigenvectors do not have to be orthogonal to each other or to eigenvectors with real energy eigenvalues.

Such non-orthogonality would have striking consequences for the behavior of unstable particles (resonances) in collisions insofar as their identity is involved. Two non-orthogonal states $|1\rangle$ and $|2\rangle$ with $\langle 1|2\rangle \neq 0$ are 'not entirely distinguishable' but have a 'partial identity' proportional to their scalar product $\langle 1|2\rangle$.

[a] e-mail: walter.blum@cern.ch
[b] e-mail: hns@mppmu.mpg.de

[1] Although we think of particles and resonances as entities to be measured and of states as representing them mathematically, we use these concepts without clearly distinguishing between them

A first category of interactions where we expect measurable effects, is quasidiffractive scattering. These processes are similar to elastic scattering like

$$\begin{aligned} \pi^- + p &\longrightarrow \pi^- + p \\ p + p &\longrightarrow p + p, \end{aligned} \tag{1.1}$$

but one of the incoming particles is excited to a short-lived state with the same charge-like quantum numbers; it subsequently decays into several decay products. Examples are

$$\begin{aligned} \pi^- + p &\longrightarrow A_1 + p \quad (A_1 \longrightarrow \rho\pi \longrightarrow \pi\pi\pi) \\ p + p &\longrightarrow N^*_{1400} + p \quad (N^*_{1400} \longrightarrow p\pi \quad \text{or} \quad \mathrm{p}\pi\pi). \end{aligned} \tag{1.2}$$

Such reactions have been measured and investigated in detail since the 1970's [1]. They have exceedingly high cross-sections which vary only slowly with the energy of the collision. This behavior is shared between the reactions of groups (1.1) and (1.2).

In order to understand their similarity we note that elastic scattering proper (1.1) is governed by the phenomenon of quantum mechanical interference of the incoming with the outgoing state, on account of their identity. We will argue that such interference is also taking place in reactions (1.2), albeit on a reduced scale; this we call the partial identity of the (incoming) particle and the (outgoing) excited state. We propose that such partial identity is intimately connected with the short lifetime of the excited state, provided its charge-like quantum numbers are the same.

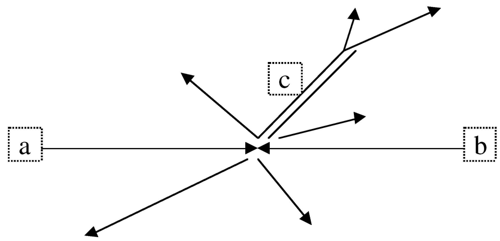


**Fig. 1.** Creation of a free unstable particle or resonance $c$ in coproduction with other particles

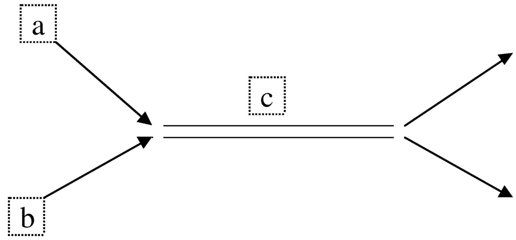


**Fig. 2.** Production of a resonance $c$ between particles $a$ and $b$ by external variation of the energy of $(a, b)$

If this is to succeed we will eventually have to include the spin of the unstable particle because quasidiffractive scattering is also observed when the excitation is to a different spin. In fact we will argue that relativistic states with different spins also have a non-zero scalar product between each other if one of them is short lived.

In the quantum mechanical formalism, an unstable decaying state ('Gamov state') is characterized by an energy width $\Gamma > 0$, leading to a damped behavior in time $\mathrm{e}^{-i(E-i\frac{\Gamma}{2})t}$. If particles and states are considered in a special relativistic framework, the energy is part of an energy-momentum vector in Minkowski space. What then happens with the energy width for a relativistic resonance, how does it have to be implemented in a Lorentz transformation compatible framework, do we have to talk about a momentum width too?

Massive stable particles are characterized by the translation and rotation eigenvalues of mass and spin. In a relativistic framework the spin determining rotation group comes as part of the Lorentz group. What about a spinning unstable particle – do we have to introduce also some sort of a spin width?

In the following we draw out the main lines of an answer to these questions. We begin with a kinematic analysis of relativistic unstable states (resonances)[2]. With respect to the possible spreads in energy, momentum and spin we employ Wigner's method of Poincaré group representations as induced by representations of the space-time translation and rotation groups. This method is prepared in Section 3 in the context of stable particles in such a way that it can be taken over and used to describe unstable particles. The first steps are done in Section 4, where we find a relativistically compatible wave function in space and time.

With it, the non-orthogonality between states one of which is unstable can be calculated for a specific Lorentz frame (Sect. 4.2). Their scalar product $\langle 1|2\rangle$ is given by their overlap integral in space; instead of a delta distribution we find a function of their momentum difference which is widened by the momentum spread. The Lorentz compatible most general case is not yet treated.

Finally, in Section 5, we describe the appearance of a phenomenon which we call 'spin spread'. The spin of a short-lived state cannot be infinitely sharp, and we present a formalism to describe the appearance of 'spin neighbors' different by 1 unit from the original spin value. This opens the possibility for two states with unequal spin to be non-orthogonal to each other.

[2] We presented momentum and spin spread on the yearly 'Workshop on Resonances and Time Asymmetric Quantum Theory' in Clausthal-Zellerfeld (Germany), 6 to 10 August, 2000 (unpublished)

## 2 Energy-momentum properties of unstable particles

The two types of reactions that can produce a free unstable particle $c$ are 'production' (2.1) and 'decay' (2.2)

$$a + b \longrightarrow c + d + e + \cdots \quad (c \longrightarrow x + y + z + \cdots) \qquad (2.1)$$
$$a \longrightarrow b + c \quad (c \longrightarrow x + y + z + \cdots) \qquad (2.2)$$

In terms of the kinematic variables of particle $c$ we may describe the most general condition of its birth by the two-body reaction

$$a + b \longrightarrow c + d \quad (c \longrightarrow x + y + z + \cdots) \qquad (2.3)$$

where $d$ represents the energy-momentum sum of all the other particles produced in the same reaction and assumed to be stable.

The situation we describe with reaction (2.3) is a free unstable particle produced as one entity (Fig. 1), in contrast to an intermediate state which is produced in parts by varying at will the energy of the incoming particles $a$ and $b$ through a resonance energy at which their cross section typically goes through a maximum (Fig. 2).

### 2.1 Centre of mass system of the production reaction

After specifying the total energy $\sqrt{s}$, the masses $m_c$ and $m_d$ as well as the decay width $\Gamma$ of particle $c$, we may calculate in the centre of mass system of reaction (2.3) the energy $E_c$ and the momentum $k_c$ of particle $c$. The conservation laws determine them to be

$$\begin{aligned} E_c &= \sqrt{s}(1 + u - v)/2 \\ k_c &= \sqrt{s}\sqrt{(1 + u - v)^2 - 4u}/2, \end{aligned} \qquad (2.4)$$

with the abbreviations $u = m_c^2/s$ and $v = m_d^2/s$. If the mass $m_c$ does not have a sharp value but is statistically distributed around its central value $m_c$ with variation

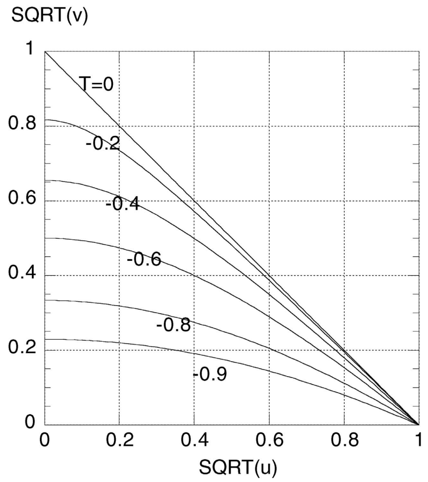


**Fig. 3.** Contours of constant $T = \delta E/\delta k$ in the $u$-$v$ plane in the centre of mass system of reaction (2.3). The physical region is the lower left hand triangle. $T$ is seen to be in the interval $-1 \leq T \leq 0$ in die entire physical region

$\pm\delta m$, the ensuing variations in $E_c$ and $k_c$ are in first order of $\delta m$ [3]

$$\delta E = \frac{\partial E}{\partial u}\frac{du}{dm}\delta m = \sqrt{u}\delta m \tag{2.5}$$

$$\delta k = \frac{\partial k}{\partial u}\frac{du}{dm}\delta m = \frac{-1+u-v}{\sqrt{(1+u-v)^2-4u}}\sqrt{u}\delta m \tag{2.6}$$

We note that the ratio $\delta E/\delta k$ is always in the interval $-1 \leq \delta E/\delta k < 0$ because the domain of $u$ and $v$ is restricted to $0 < u < 1, \quad 0 \leq v < 1, \quad |\sqrt{u}| + |\sqrt{v}| < 1$.

In Fig. 3 the $u$-$v$-plane is shown with contours of constant $\delta E/\delta k$, also called $T$, derived from (2.5), (2.6). It is interesting to note that the value of $T$, which describes how the uncertainty of $m$ propagates into energy and momentum, depends on the overall reaction as well as on the unstable particle itself. It is easy to see from (2.4) to (2.6) that $T$ is equal to the velocity of particle $d$, the energy-momentum sum of all the reaction partners of the unstable particle $c$. This holds for the moment in the centre of mass of the production reaction but can be generalized.

For the exponential decay a Breit-Wigner form for $m_c$ is appropriate, and the width of the distribution is given by $\delta m = \Gamma/2$. The corresponding widths of $E$ and $k$ are

$$\Delta E = \sqrt{u}\Gamma/2 \tag{2.7}$$

$$\Delta k = \frac{-1+u-v}{\sqrt{(1+u-v)^2-4u}}\sqrt{u}\Gamma/2. \tag{2.8}$$

We call them energy- and momentum-spread.

## 2.2 Lorentz-boost along the direction of motion

If a particle characterized by $(E, k, \Delta E, \Delta k)$ is Lorentz-boosted along its direction of motion, using the velocity $\beta$ and $\gamma^2 = 1/(1-\beta^2)$, the values in the new system are given by

$$\begin{pmatrix} E' \\ k' \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} E \\ k \end{pmatrix} \tag{2.9}$$

$$\begin{pmatrix} \Delta E' \\ \Delta k' \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \Delta E \\ \Delta k \end{pmatrix} \tag{2.10}$$

The widths $(\Delta E, \Delta k)$ transform exactly as the energy-momentum $(E, k)$. The variable $T = \Delta E/\Delta k$ transforms as

$$T' = \frac{T+\beta}{T\beta+1}, \tag{2.11}$$

it can reach all values in the interval $-1 \leq T < 1$ even though it was restricted to the interval $-1 \leq T < 0$ in the centre of mass system of reaction (2.3).

The invariants of the Lorentz-boost are

$$\begin{aligned} (E,k)^2 &= \quad E^2 - k^2 = m^2 \\ (\Delta E, \Delta k)^2 &= (\Delta E)^2 - (\Delta k)^2 = -B^2 \\ (E,k)(\Delta E, \Delta k) &= \ E\Delta E - k\Delta k = m\Gamma/2 \end{aligned} \tag{2.12}$$

The third relation in (2.12) is best evaluated in the rest system of the unstable particle where $k = 0$ and $E = m$. In the second relation we have introduced $B$ (capital beta); the decaying state is characterized by both $\Gamma$ and $B$. It turns out that $(\Delta E, \Delta k)$ is spacelike. One verifies this in the centre of mass system of reaction (2.3):

$$\begin{aligned} -B^2 &= (\Delta E)^2 - (\Delta k)^2 \\ &= \frac{-4uv}{(1+u-v)^2-4u}\frac{\Gamma^2}{4} \leq 0 \end{aligned} \tag{2.13}$$

In a diagram $E$ *vs.* $k$ a stable particle with rest mass $m$ is located on the hyperbola $E = \sqrt{(k^2+m^2)}$ through $E = m$ at $k = 0$ (Fig. 4). If this particle is unstable so that its mass is distributed around $m$ with width $\Gamma$ one may draw two additional hyperbolas through $E = m \pm \Gamma/2$ at $k = 0$. The unstable state is characterized by a small vector $(\Delta E, \Delta k)$ at $(E, k)$ connecting the two outer hyperbolas, the slope not exceeding 1. This restriction is a consequence of the spacelike nature of $(\Delta E, \Delta k)$. The unstable particle is measured as a statistical ensemble of individual events. Each individual event is represented as a point along the small vector. There is a complete correlation between the deviations from the mean in $E$ and the deviations from the mean in $k$.

## 2.3 Interpretation and examples

In the rest system (*) of the unstable particle where $E^* = m, k^* = 0, \Delta E^* = \Gamma/2$ this means especially that $|\Delta k^*|$ is at least as large as $\Gamma/2$. The different parts of the mass distribution are not all at rest; when the centre is at rest

[3] The index c is omitted from here onwards

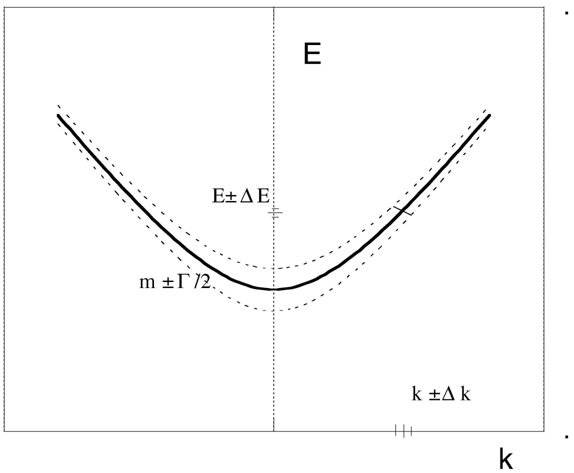


**Fig. 4.** An unstable particle in the energy-momentum plane is characterized by a straight line through the point $(E,k)$ on the hyperbola $(m)$, the relation between $\Delta E$, $\Delta k$ and $\Gamma$ is displayed

**Table 1.** Values assumed by the two 4-vectors in the two special Lorentz frames

| 4-vector | Central rest system | Sharp energy system |
|---|---|---|
| $(E,k)$ | $(m,0)$ | $m\left(\sqrt{1+\frac{\Gamma^2}{4B^2}},\frac{\Gamma}{2B}\right)$ |
| $(\Delta E,\Delta k)$ | $\frac{\Gamma}{2}\left(1,-\sqrt{1+\frac{4B^2}{\Gamma^2}}\right)$ | $(0,-B)$ |

the two wings fly apart in opposite directions. One may say that an unstable particle cannot come entirely to rest. The consequences for the spin of unstable particles are discussed further down. The velocities involved are of the order of $\Gamma/m$ times the speed of light, which can be read off Fig. 4.

The value of $\Delta k^*$ can also be calculated directly from (2.7), (2.8) and (2.10) by applying a boost back into the rest system of the unstable particle (velocity $\beta=-k/E$, (2.4)):

$$\Delta k^* = \beta\gamma\Delta E+\gamma\Delta k = -\frac{k}{m}\Delta E+\frac{E}{m}\Delta k \quad (2.14)$$

$$= \frac{-1+u+v}{\sqrt{(1+u-v)^2-4u}}\,\frac{\Gamma}{2} \quad (2.15)$$

The existence of the two fourvectors gives rise to two special Lorentz frames, one in which the average momentum $k=0$, we call it 'the central rest system' of the unstable particle, and one in which $\Delta E=0$; this we call 'the sharp energy system' of the unstable particle. Here we have added 'central' to the familiar name of 'rest system' because of the peculiar situation that the particles in the statistical ensemble that makes up the resonance can not all be simultaneously at rest. Table 1 summarizes the values of the two 4-vectors in these two systems.

The following examples illustrate the different values of $\Delta E/\Delta k$ and $B^2$ as functions of the way the unstable state is produced. A $\rho$-meson ($m=0.77$ GeV/c$^2$, $\Gamma=$ 0.15 GeV/c$^2$) is produced by a $\pi$-meson beam on a hydrogen target in the forward direction: $\pi+p\longrightarrow\rho+p$. In a first case the beam momentum $k_{beam}$ is 2.5 GeV/c, in a second case $k_{beam}=100$ GeV/c. The third case is a stable $D^+$(1869 MeV/c$^2$) and an unstable $D^{*-}$(2460 MeV/c$^2$, $\Gamma=25$ MeV/c$^2$), produced as a pair in e$^+$e$^-$ annihilation at $\sqrt{s}=5$ GeV/c$^2$. We quote in Table 2 numerical values of relevant variables for these cases.

**Table 2.** Numerical values of some relevant variables for the three examples quoted in the text

| Variable | Case 1 | Case 2 | Case 3 |
|---|---|---|---|
| $k_{beam}$ (GeV/c) | 2.5 | 100 | $2\times 2.5$ |
| $\sqrt{s}$ (GeV/c$^2$) | 2.37 | 13.73 | 5.00 |
| $u$ | 0.1059 | $3.144\cdot 10^{-3}$ | 0.242 |
| $v$ | 0.1676 | $4.669\cdot 10^{-3}$ | 0.140 |
| CM system of the reaction: | | | |
| $E$ (GeV/c$^2$) | 1.1101 | 6.855 | 2.755 |
| $k$ (GeV/c) | 0.7969 | 6.812 | 1.241 |
| $\Delta E/(\Gamma/2)$ | 0.325 | 0.0561 | 0.492 |
| $\Delta k/(\Gamma/2)$ | −0.511 | −0.0568 | −0.890 |
| CM system of the unstable particle: | | | |
| $\Delta k^*/(\Gamma/2)$ | -1.075 | -1.000030 | -1.245 |
| Invariant: | | | |
| $B/(\Gamma/2)$ | 0.3943 | 0.00892 | 0.7416 |

Another possible reaction is a nucleus decaying from some metastable state 1 into a state 2 with very short lifetime $(1/\Gamma)$ by the emission of a light particle. If one identifies $\sqrt{s}$ with the mass of state 1, $u$ with the square of the mass of state 2 over $s$ ($u$ close to 1), and neglecting the mass of the light particle ($v=0$) we find $B^2=0$ and $\Delta k^*=-\Gamma/2$ using (2.13) and (2.15).

## 2.4 Lorentz-boost in an arbitrary direction

Up to here the uncertainty $\Delta k$ in momentum was along the direction of motion. Now we want to apply a Lorentz transformation with velocity $\vec{\beta}$ along a different, arbitrary direction. Let $\vec{\beta}$ have an angle $\theta$ with $\vec{k}$ and let $\vec{\Delta k}$ be parallel to $\vec{k}$. Expressing the components along $\vec{\beta}$ by $k\cos\theta$ and $\Delta k\cos\theta$, and the components transverse to $\vec{\beta}$ by $k\sin\theta$ and $\Delta k\sin\theta$, the kinematic variables of the unstable particle $E,\vec{k},\Delta E,\vec{\Delta k}=(\Delta k/k)\vec{k}$ transform from the rest system of (2.3) into the primed system as follows:

$$\begin{aligned} E' &= \gamma E+\beta\gamma k\cos\theta; \\ k'\cos\theta' &= \beta\gamma E+\gamma k\cos\theta; \quad (2.16) \\ k'\sin\theta' &= k\sin\theta; \\ \Delta E' &= \gamma\Delta E+\beta\gamma\Delta k\cos\theta; \\ \Delta k'\cos\Theta' &= \beta\gamma\Delta E+\gamma\Delta k\cos\theta; \quad (2.17) \end{aligned}$$

$$\Delta k' \sin \Theta' = \Delta k \sin \theta,$$

where $\beta = |\vec{\beta}|$ and $\gamma^2 = 1/(1-\beta^2)$. In the primed system, the angle of the 3-momentum vector $\vec{k'}$ is denoted by $\theta'$, whereas the angle of the momentum uncertainty $\vec{\Delta k'}$ against $\vec{\beta}$ is denoted by $\Theta'$. These two angles will be different unless $\sin\theta$ vanishes: dividing the last two equations in each group, the transformed angles are given by

$$\tan \theta' = \frac{1}{\gamma} \frac{\sin\theta}{\beta \frac{E}{k} + \cos\theta} \tag{2.18}$$

$$\tan \Theta' = \frac{1}{\gamma} \frac{\sin\theta}{\beta \frac{\Delta E}{\Delta k} + \cos\theta} \tag{2.19}$$

These two expressions are not the same because the ratio $E/k$ of the timelike vector $(E, \vec{k})$ is $> 1$ whereas the ratio $\Delta E/\Delta k$ of the spacelike vector $(\Delta E, \vec{\Delta k})$ is $< 1$. We conclude that in the general case the two vectors $\vec{k'}$ and $\vec{\Delta k'}$ will no longer be parallel but may in principle have any angle between them.

The situation described with the two variables $E$ and $\Delta k$ in Sects. 2.1, 2.2 and 2.3 reveals itself as a special case of a 4-vector $(\Delta E, \vec{\Delta k})$ where $\vec{\Delta k}$ remained parallel to $\vec{k}$, as long as one stayed in a special set of Lorentz systems. These are defined as being connected to the centre-of-mass system of the production reaction by a boost along the direction of motion of the unstable particle.

The most general case of a spread vector $(\Delta E, \vec{\Delta k})$ is given by 4 numbers which describe the complete correlation between the deviation of the individual particles of the ensemble from their mean values of $E, k_x, k_y$, and $k_z$.

## 3 Stable states

In this section we determine the relativistically compatible wave function for a stable particle from its Feynman propagator.

### 3.1 Wave functions for stable scattering states in quantum mechanics

The time behavior of a stable energy eigenstate in quantum mechanics is described by a $\mathbf{U}(1)$-representation of the time translations $\mathbb{R}$

$$\begin{array}{l} \text{representation: } \mathbb{R} \ni t \longmapsto \mathrm{e}^{-iEt} \in \mathbf{U}(1) \\ \psi(t) = \mathrm{e}^{-iEt}|\psi\rangle \text{ with energy } E \in \mathbb{R} \end{array} \tag{3.1}$$

The free scattering states with momentum $\vec{k}$ can be built - with the position space orbits in the Schrödinger picture $|\psi\rangle \cong \psi(\vec{x})$ - by plane waves $\mathrm{e}^{-i\vec{k}\vec{x}}$ which, by themselves, are no Hilbert states - only their packets $\int d^3k \mathrm{e}^{-i\vec{k}\vec{x}}\mu(\vec{k})$ which use square integrable momentum functions $\int d^3k|\mu(\vec{k})|^2 < \infty$. For a constant potential $V_0$, energy and the momentum of a scattering state are related as follows

$$\frac{\vec{k}^2}{2} = E - V_0 \tag{3.2}$$

In polar coordinates for a radial symmetric dynamics (angular momentum invariant Hamiltonian $[\vec{\mathcal{L}}, H] = 0$) the scattering states are built by packets of spherical Bessel waves $j_L$ for the Hilbert space with the radial translations $r \in \mathbb{R}^+$ functions for each angular momentum eigenvalue $L = 0, 1, \ldots$. The spherical Bessel functions [6] are definable as plane wave coefficients with respect to the sperical harmonics $\mathrm{Y}_0^L(\theta,\varphi) = \sqrt{\frac{1+2L}{4\pi}}\mathrm{P}^L(\cos\theta)$

$$\begin{aligned} \mathrm{e}^{i\vec{k}\vec{x}} &= \mathrm{e}^{ikr\cos\theta} = \mathrm{e}^{iR\zeta} : j_L(R) = \frac{1}{2i^L}\int_{-1}^{1} d\zeta\ \mathrm{P}^L(\zeta)\mathrm{e}^{iR\zeta} \\ &= R^L(-\frac{1}{R}\frac{d}{dR})^L \frac{\sin R}{R} \\ &\Rightarrow \mathrm{e}^{i\vec{q}\vec{x}} = \sum_{L=0}^{\infty} i^L j_L(qr)\ (1+2L)\mathrm{P}^L(\cos\theta) \end{aligned} \tag{3.3}$$

Time reversal, implemented by $t \leftrightarrow -t$ and number conjugation $\alpha \leftrightarrow \overline{\alpha}$, relate to each other the conjugated Hankel waves in the standing Bessel waves

$$j_L = \frac{h_L^+ - h_L^-}{2i},\ h_L^{\pm}(R) = R^L(-\tfrac{1}{R}\tfrac{d}{dR})^L \frac{\mathrm{e}^{\pm iR}}{R} \tag{3.4}$$

They are the in- and outcoming waves with the large distance behavior reflecting the large time (future and past) behavior

$$R \to \infty : \begin{cases} j_L(R) \to \frac{\sin(R - \frac{L\pi}{2})}{R}, \\ \qquad \text{standing waves} \\ h_L^{\pm}(R) \to (\mp i)^L \frac{\mathrm{e}^{\pm iR}}{R}, \\ \qquad \text{in- and outgoing waves} \end{cases} \tag{3.5}$$

### 3.2 Energy-momentum and spin in Minkowski space

Special relativity is characterized by the Poincaré group $\mathbf{SO}_0(1,3) \vec{\times}\, \mathbb{R}^4$ as semidirect product of the orthochronous Lorentz group $\mathbf{SO}_0(1,3)$ acting on the spacetime translations (Minkowski space) $x \in \mathbb{R}^4$ and on its dual space, the translation eigenvalues $q \in \mathbb{R}^4$, which constitute the energy-momentum space.

As first realized by Wigner, particles can be classified according to the stability group for their causal energy-momenta, i.e. for $q \in \mathbb{R}^4$ with $q^2 = m^2 \geq 0$. Strictly positive mass particles, $m^2 > 0$, e.g. electrons, have transformation properties with respect to spin $\mathbf{SU}(2)$. Massless particles, $m^2 = 0$, e.g. photons, are characterized with respect to the spin subgroup polarization $\mathbf{SO}(2)$.

In the following we will restrict ourselves to massive particles, i.e. to an embedding of the spin group $\mathbf{SU}(2)$, the double cover[4] of the rotation group $\mathbf{SO}(3) \cong \mathbf{SU}(2)/\{\pm\mathbf{1}_2\}$, into the real 6-dimensional group $\mathbf{SL}(\mathbb{C}^2)$, the double cover of the orthochronous Lorentz group $\mathbf{SO}_0(1,3) \cong \mathbf{SL}(\mathbb{C}^2)/\{\pm\mathbf{1}_2\}$. For a relativistically compatible description, the nonrelativistic direct product group

[4] In the following, both $\mathbf{SO}(3)$ and $\mathbf{SU}(2)$ will be called - somewhat sloppily - both spin and rotation group, and both $\mathbf{SO}_0(1,3)$ and $\mathbf{SL}(\mathbb{C}^2)$ come under the name of Lorentz group

with the rotations and the time translations is embedded [5] as a subgroup of the semidirect Poincaré group

$$\begin{aligned} \mathbf{SO}(3)\times\mathbb{R} &\hookrightarrow \mathbf{SO}_0(1,3)\,\overrightarrow{\times}\,\mathbb{R}^4 \\ \mathbf{SU}(2)\times\mathbb{R} &\hookrightarrow \mathbf{SL}(\mathbb{C}^2)\,\overrightarrow{\times}\,\mathbb{R}^4 \end{aligned} \tag{3.6}$$

With respect to the translation subgroups, the unitary representations of the time translations are characterized by energies $E\in\mathbb{R}$, and the ones of spacetime translations by energy-momenta $q\in\mathbb{R}^4$:

$$\mathbb{R}\ni t\longmapsto e^{-iEt}\in\mathbf{U}(1),\ \mathbb{R}^4\ni x\longmapsto e^{-iqx}\in\mathbf{U}(1) \tag{3.7}$$

With respect to the homogeneous subgroups: Halfinteger and integer spin numbers $S$ characterize the irreducible unitary representation of the spin group

$$D^S:\mathbf{SU}(2)\longrightarrow\mathbf{SU}(1+2S)\subset\mathbf{GL}(\mathbb{C}^{1+2S}), \tag{3.8}$$

whereas the finite dimensional irreducible representations of the Lorentz group - which, if nontrivial, are non-unitary - come with halfinteger and integer 'left-right spin' numbers $[L|R]$

$$\begin{aligned} &D^{[L|R]}:\mathbf{SL}(\mathbb{C}^2) \\ &\quad\longrightarrow\mathbf{SL}(\mathbb{C}^{(1+2L)(1+2R)})\subset\mathbf{GL}(\mathbb{C}^{(1+2L)(1+2R)}) \end{aligned} \tag{3.9}$$

The relativistically compatible embedding of a stable particle with spin-mass $(S,m)$, characterized in its rest system by a spin $\mathbf{SU}(2)$ and time translation $\mathbb{R}$ representation in the unitary group $\mathbf{U}(1)\circ\mathbf{SU}(1+2S)$ of a complex $(1+2S)$-dimensional vector space

$$\begin{aligned} \mathbf{SU}(2)\times\mathbb{R}\ni(u=e^{i\vec{\alpha}\vec{\sigma}/2},t)& \\ \longmapsto D^S(u)\times e^{-imt}&\in\mathbf{U}(1+2S) \end{aligned} \tag{3.10}$$

is given by

$$\begin{aligned} &\mathbf{SU}(2)\times\mathbb{R}\ni(S,m)\hookrightarrow([L|R];q)\in\mathbf{SL}(\mathbb{C}^2)\,\overrightarrow{\times}\,\mathbb{R}^4 \\ &S,L,R\in\{0,\tfrac{1}{2},1,\dots),\ q^2=m^2>0. \end{aligned} \tag{3.11}$$

The relation of spin $S$ to Lorentz $\{L,R\}$ will be discussed below.

### 3.3 Representations of spacetime translations

The embedding of the unitary representations of the time translations for an energy $E^2=m^2$ into a the unitary representations of the spacetime translations for energy-momentum $q^2=m^2$ is characterizable with the generalized functions

$$\delta(E^2-m^2)\hookrightarrow\delta(q^2-m^2) \tag{3.12}$$

The unitary time translation matrix elements from $t\longmapsto e^{itm}\in\mathbf{U}(1)$ can be written as supported by an energy distribution

$$\begin{aligned} \begin{pmatrix}\cos tm\\ -i\sin tm\end{pmatrix} &= \int dE\ \epsilon(m)\begin{pmatrix}m\\ E\end{pmatrix}\delta(E^2-m^2)e^{-tiE} \\ &= \int dE\ \epsilon(E)\begin{pmatrix}E\\ m\end{pmatrix}\delta(E^2-m^2)e^{-tiE} \\ &= -\epsilon(t)\int\frac{dE}{i\pi}\frac{1}{E_{\mathrm{P}}^2-m^2}\begin{pmatrix}E\\ m\end{pmatrix}e^{-tiE} \end{aligned} \tag{3.13}$$

Here complex distributions are involved with a Dirac function as real part and a principal value distribution as imaginary part (integration with positive $o$, then limit $o\to0$)

$$a\in\mathbb{R},\ \ \pm\frac{1}{i\pi}\frac{1}{a\mp io}=\delta(a)\pm\frac{1}{i\pi}\frac{1}{a_{\mathrm{P}}} \tag{3.14}$$

and the step functions

$$a\in\mathbb{R},\ \begin{cases}\vartheta(a)+\vartheta(-a)=1\\ \vartheta(a)-\vartheta(-a)=\epsilon(a)=\frac{a}{|a|}\end{cases} \tag{3.15}$$

The time representation matrix elements are embeddable with an energy-momentum mass hyperboloid

$$\mathbb{R}\ni m\hookrightarrow q=(q_j)_{j=0,1,2,3}\in\mathbb{R}^4\text{ with }q^2=m^2 \tag{3.16}$$

in two ways

$$\begin{aligned} d_t\begin{pmatrix}\cos tm\\ \sin tm\end{pmatrix}&=m\begin{pmatrix}-\sin tm\\ \cos tm\end{pmatrix} \\ \hookrightarrow&\begin{cases}\partial^j\begin{pmatrix}\mathbf{C}(x|m)\\ \mathbf{S}_j(x|m)\end{pmatrix}=m\begin{pmatrix}-\mathbf{S}^j(x|m)\\ \mathbf{C}(x|m)\end{pmatrix}\\ \partial^j\begin{pmatrix}\mathbf{c}_j(x|m)\\ \mathbf{s}(x|m)\end{pmatrix}=m\begin{pmatrix}-\mathbf{s}(x|m)\\ \mathbf{c}^j(x|m)\end{pmatrix}\end{cases} \end{aligned} \tag{3.17}$$

The boldface symbols with two arguments for translations and eigenvalue, e.g. $\mathbf{C}(x|m)$, embed the trigonometric functions with the corresponding notation for time translations and energy eigenvalue, e.g. $\cos tm$.

Both embeddings come with a Lorentz scalar and a Lorentz vector: One embedding (Lorentz scalar cosine and vector sine) involves the functions

$$\begin{pmatrix}\mathbf{C}(x|m)\\ -i\mathbf{S}_j(x|m)\end{pmatrix}=\int\frac{d^4q}{(2\pi)^3}\epsilon(m)\begin{pmatrix}m\\ q_j\end{pmatrix}\delta(q^2-m^2)e^{-iqx} \tag{3.18}$$

which occur as Fock state functions for relativistic particle fields (next subsection). The other embedding (Lorentz scalar sine and vector cosine) with an ordered Dirac energy-momentum measure

$$\begin{aligned} \begin{pmatrix}\mathbf{c}_j(x|m)\\ -i\mathbf{s}(x|m)\end{pmatrix}&=\int\frac{d^4q}{(2\pi)^3}\epsilon(q_0)\begin{pmatrix}q_j\\ m\end{pmatrix}\delta(q^2-m^2)e^{-iqx} \\ &=-\epsilon(x_0)\int\frac{d^4q}{i\pi(2\pi)^3}\begin{pmatrix}q_j\\ m\end{pmatrix}\frac{1}{q_{\mathrm{P}}^2-m^2}e^{-iqx} \end{aligned} \tag{3.19}$$

defines distributions which occur for the relativistic field quantization. The ordered Lebesque measure $d^4q\epsilon(q_0)\vartheta(q^2)$ leads to causal support

$$\begin{pmatrix}\mathbf{c}_j(x|m)\\ \mathbf{s}(x|m)\end{pmatrix}=0\text{ for }x^2<0 \tag{3.20}$$

All those embeddings, $(\mathbf{C}, \mathbf{S}_j)$ and $(\mathbf{c}_j, \mathbf{s})$, are 'on-shell', i.e. energy-momentum supported by $q_0^2 - \vec{q}^{\,2} = m^2$. They are matrix elements of spacetime translation representations in unitary Poincaré group representations.

The cross over sums are used in Feynman propagators for relativistic quantum particle fields and embed the causal time representations

$$\begin{aligned}
\mathrm{e}^{\mp i|tm|} &= \mathbf{C}(x|m) \mp \epsilon(x_0 m) i\mathbf{s}(x|m) \\
&= \int \tfrac{d^4q}{(2\pi)^3} \vartheta(\pm x_0 q_0) 2|m| \delta(q^2 - m^2) \mathrm{e}^{-iqx} \\
&= \mp \int \tfrac{d^4q}{i\pi(2\pi)^3} \tfrac{|m|}{q^2 \pm io - m^2} \mathrm{e}^{-iqx}
\end{aligned} \tag{3.21}$$

As visible in the last line, Feynman propagators are supported also 'off-shell', i.e. for $q_0^2 - \vec{q}^{\,2} \neq m^2$ ('virtual particles').

The embedding of the nonrelativistic in- and outgoing wave functions of the foregoing subsection

$$\begin{aligned}
\psi_L^{\pm}(t, \vec{x}) &= \mathrm{e}^{-iEt} k h_L^{\pm}(kr) \mathrm{Y}_m^L(\theta, \varphi), \\
&\quad \text{with } k = \sqrt{2(E - V_0)} \\
\text{e.g. } \psi_0^{\pm}(t, \vec{x}) &= \mathrm{e}^{-iEt} \frac{\mathrm{e}^{\pm ikr}}{r}, \ \psi_1^{\pm}(t, \vec{x}) \\
&= \mathrm{e}^{-iEt} \frac{1 \mp ikr}{kr} \frac{\mathrm{e}^{\pm ikr}}{r} \\
&\quad \times \sqrt{\frac{3}{4\pi}} \begin{pmatrix} \mp \frac{1}{\sqrt{2}} \sin\theta \, \mathrm{e}^{\pm i\varphi} \\ \cos\theta \end{pmatrix}, \ldots
\end{aligned} \tag{3.22}$$

is seen explicitly in the harmonic analysis with respect to time and position space translations. In the scalar cosine of Minkowski spacetime the time representations come with standing $L = 0$ spherical waves as position realizations, if the energy $q_0$ surpasses the mass threshold $m$

$$\begin{aligned}
\frac{\mathbf{C}(x|m)}{|m|} &= \int \frac{d^4q}{(2\pi)^3} \delta(q^2 - m^2) \mathrm{e}^{-iqx} \\
&= -\frac{1}{r}\frac{d}{dr} \int \frac{d^2q}{(2\pi)^2} \delta(q^2 - m^2) \mathrm{e}^{-iqx}|_{x=(t,r)} \\
&= \int \frac{dq_0}{(2\pi)^2} \vartheta(q_0^2 - m^2) \mathrm{e}^{-iq_0 t} \frac{\sin r\sqrt{q_0^2 - m^2}}{r}
\end{aligned} \tag{3.23}$$

For Lorentz scalar integrands, the 2-sphere integration (polar coordinates with $r = |\vec{x}|$) over the 2-sphere $\mathbf{SO}(3)/\mathbf{SO}(2)$ goes over from the Lorentz group $\mathbf{SO}_0(1,3)$ to an abelian noncompact subgroup $\mathbf{SO}_0(1,1)$ with trivial spin. It yields the characteristic 2-sphere distribution factor $\frac{1}{r}$ (Kepler factor). For the Lorentz group $\mathbf{SO}_0(1,1)$ in two spacetime dimensions the integrals $d^2q = dq_0 dq_3$ both over the energy $q_0$ and the directed momentum modulus $q_3 = \epsilon(q_3)|\vec{q}|$ go over the full real axis $\int_{-\infty}^{\infty}$.

The harmonic analysis of both the scalar sine and the ordered scalar sine displays irreducible time representations multiplied with spherical waves for energies $q_0^2$ over the threshold $m^2$ (on shell). The off shell contributions of the ordered sine for $q_0^2$ smaller than $m^2$ give irreducible time representations, multiplied with a Yukawa potential

$$\begin{aligned}
&\begin{pmatrix} 1 \\ \epsilon(x_0) \end{pmatrix} \frac{i\mathbf{s}(x|m)}{m} \\
&= -\int \frac{d^4q}{(2\pi)^3} \begin{pmatrix} \epsilon(q_0)\delta(q^2 - m^2) \\ -\frac{i}{\pi} \frac{1}{-q_{\mathrm{P}}^2 + m^2} \end{pmatrix} \mathrm{e}^{-iqx} \\
&= \frac{1}{r}\frac{d}{dr} \int \frac{d^2q}{(2\pi)^2} \begin{pmatrix} \epsilon(q_0)\delta(q^2 - m^2) \\ -\frac{i}{\pi} \frac{1}{-q_{\mathrm{P}}^2 + m^2} \end{pmatrix} \mathrm{e}^{-iqx}|_{x=(t,r)} \\
&= \int \frac{dq_0}{i(2\pi)^2} \mathrm{e}^{-iq_0 t} \left[ \vartheta(q_0^2 - m^2) \begin{pmatrix} -\epsilon(q_0) i \frac{\sin r\sqrt{q_0^2 - m^2}}{r} \\ \frac{\cos r\sqrt{q_0^2 - m^2}}{r} \end{pmatrix} \right. \\
&\quad \left. + \vartheta(m^2 - q_0^2) \begin{pmatrix} 0 \\ \frac{\mathrm{e}^{-r\sqrt{m^2 - q_0^2}}}{r} \end{pmatrix} \right]
\end{aligned} \tag{3.24}$$

The harmonic analysis of the Feynman propagator contains the sum of an 'on shell' particle with spherical wave, in- or outgoing as determined by $\pm io$ resp., and an 'off shell' Yukawa interaction

$$\begin{aligned}
&\mp \int \frac{d^4q}{i\pi(2\pi)^3} \frac{1}{q^2 \pm io - m^2} \mathrm{e}^{-iqx} \\
&= \pm \frac{1}{r}\frac{d}{dr} \int \frac{d^2q}{i\pi(2\pi)^2} \frac{1}{q^2 \pm io - m^2} \mathrm{e}^{-iqx}|_{x=(t,r)} \\
&= \pm \int \frac{dq_0}{i(2\pi)^2} \mathrm{e}^{-iq_0 t} \left[ \vartheta(q_0^2 - m^2) \frac{\mathrm{e}^{\pm ir\sqrt{q_0^2 - m^2}}}{r} \right. \\
&\quad \left. + \vartheta(m^2 - q_0^2) \frac{\mathrm{e}^{-r\sqrt{m^2 - q_0^2}}}{r} \right]
\end{aligned} \tag{3.25}$$

### 3.4 Relativistic wave functions for spinless stable particles

The matrix elements of unitary spacetime translation representations of the foregoing subsection arise for a massive spinless particle, e.g. the $\pi^0$-meson, considered as stable.

A relativistic hermitian scalar Bose field $\mathbf{A}(x)$ for such a particle has the harmonic decomposition into translation eigenvectors, called creation operators $\mathrm{u}(\vec{q})$ and annihilation operators $\mathrm{u}^{\star}(\vec{q})$ for momentum $\vec{q}$

$$\begin{aligned}
\mathbf{A}(x) &= \int \tfrac{d^3q}{q_0(2\pi)^3} \tfrac{\mathrm{e}^{-iqx}\mathrm{u}(\vec{q}) + \mathrm{e}^{iqx}\mathrm{u}^{\star}(\vec{q})}{\sqrt{2}} \\
&\text{with } q = (q_0, \vec{q}), \ q_0 = \sqrt{m^2 + \vec{q}^{\,2}}
\end{aligned} \tag{3.26}$$

As indexed by the momenta $\vec{q} \in \mathbb{R}^3$, the associate state space is overcountably infinite dimensional.

The basic vectors have as commutators (quantization) and as Fock state value for anticommutators (denoted with $\langle \ldots \rangle_{\mathrm{F}}$)

$$\begin{aligned}
[\mathrm{u}^{\star}(\vec{p}), \mathrm{u}(\vec{q})] &= (2\pi)^3 q_0 \delta(\vec{q} - \vec{p}) \\
\langle \{\mathrm{u}^{\star}(\vec{p}), \mathrm{u}(\vec{q})\} \rangle_{\mathrm{F}} = \langle \mathrm{u}(\vec{p}) | \mathrm{u}(\vec{q}) \rangle &= (2\pi)^3 q_0 \delta(\vec{q} - \vec{p})
\end{aligned} \tag{3.27}$$

With the shorthand notation for translation dependent (anti) commutators of spacetime dependent operators

$$\begin{aligned}
\epsilon = \pm 1: \ [A, B]_{\epsilon}(x) &= [A(x_2), B(x_1)]_{\epsilon} \\
&= A(x_2)B(x_1) + \epsilon B(x_1)A(x_2) \\
&\quad \text{for all } x = x_1 - x_2
\end{aligned} \tag{3.28}$$

the Feynman propagator sums up the on-shell Fock value of the quantization opposite anticommutator and the on- and off-shell causally ordered quantization commutator

$$\begin{aligned}\langle\{\mathbf{A},\mathbf{A}\}(x)-\epsilon(x_0)[\mathbf{A},\mathbf{A}](x)\rangle_{\mathrm{F}}\\ =\frac{\mathbf{C}(x|m)-\epsilon(x_0)i\mathbf{s}(x|m)}{m}\\ =-\int\frac{d^4q}{i\pi(2\pi)^3}\frac{1}{q^2+io-m^2}\mathrm{e}^{iqx}\end{aligned}\qquad(3.29)$$

The wave function for an outgoing scalar particle (denoted with the corresponding non-boldface letter – here $A$ for the particle field $\mathbf{A}$), considered from a reference system wherein the particle has energy $q_0=E>m$ can be obtained by Dirac-picking with $\delta(q_0-E)$ in the harmonic expansion of the Feynman propagator above the corresponding contribution with the angular momentum $L=0$ outgoing Hankel wave (an irrelevant normalization factor $\frac{i}{(2\pi)^2}$ is omitted)

$$\begin{aligned}A(t,r)&=-\int\frac{d^4q}{2\pi^2}\frac{1}{q^2+io-m^2}\mathrm{e}^{-iqx}\delta(q_0-E)\\ &=\mathrm{e}^{-iEt}\frac{\mathrm{e}^{ikr}}{r}\\ &=\mathrm{e}^{-iEt}\ kh_0^+(kr)\\ &\quad\text{with } k=\sqrt{E^2-m^2}\end{aligned}\qquad(3.30)$$

Summarizing, the relativistic wave function for an outgoing scalar particle is

$$\boxed{A(t,r)=\mathrm{e}^{-iEt}\ \frac{\mathrm{e}^{ikr}}{r}}\qquad(3.31)$$

In the rest system of the particle, denoted by the supscript $\overset{\mathrm{rs}}{=}$, one obtains the product of the time orbit with the Kepler 2-sphere distribution factor

$$(E,k)\overset{\mathrm{rs}}{=}(m,0)\Rightarrow A(t,r)\overset{\mathrm{rs}}{=}\frac{\mathrm{e}^{-imt}}{r}\qquad(3.32)$$

## 4 Unstable states

### 4.1 Construction of the wave function

Our starting point is (3.30) where the wave function for a stable particle of a given energy $E>m$ was obtained from the harmonic expansion of the Feynman propagator: From the reservoir of all wave functions $\mathrm{e}^{-iqx}$ one picks the one with energy $E$.

For an unstable particle we introduce two modifications: Firstly the distributional 'io' in (3.30) is replaced by the invariant width '$im\Gamma$'. Obviously the stable particle results have to reappear in the limit $\Gamma\to 0$.

Secondly, $q_0$ is no longer fixed to be equal to $E$; the integration variables must be correlated in a different way. In the simple case treated in Sects. 3.1 to 3.3 ($\overrightarrow{\Delta k}$ parallel to $\overrightarrow{k}$), both $k\equiv|\overrightarrow{k}|$ and $\Delta k\equiv|\overrightarrow{\Delta k}|$ are independent of direction, and the correlation is given by the expression

$$\frac{q_0-E}{|\overrightarrow{q}|-k}=\frac{\Delta E}{\Delta k}$$

Therefore we replace in (3.30)

$$\delta(q_0-E)\hookrightarrow\delta(C(q_0-E)-S(|\overrightarrow{q}|-k))\qquad(4.1)$$

where

$$C=\pm\Delta k/\sqrt{(\Delta k)^2-(\Delta E)^2}$$

$$S=\pm\Delta E/\sqrt{(\Delta k)^2-(\Delta E)^2}\qquad(4.2)$$

The normalization factor $\sqrt{(\Delta k)^2-(\Delta E)^2}$, or $B$ (cf. (2.12)) was introduced because it guarantees that the case of $\Delta E=0$ falls back into the old form $\delta(q_0-E)$, and because it is Lorentz-invariant. The signs in (4.2) must be chosen such that $C>0$.

Looking at the new $\delta$-distribution (4.1), we observe that its argument may be interpreted as the energy component of a vector in an energy-momentum space spanned by the deviations $(q_0-E)$ and $(|\overrightarrow{q}|-k)$. In fact, the normalized coefficients $C$ and $S$ may be interpreted as coefficients of a Lorentz boost

$$C=\cosh\zeta,\ \ S=\sinh\zeta\qquad(4.3)$$

from the general system in which the decaying particle has $(E,k)$ and $(\Delta E,\Delta k)$ to the sharp energy system (primed variables) in which $\Delta E'=0$:

$$\begin{pmatrix}a'\\ b'\end{pmatrix}=\begin{pmatrix}C&-S\\ -S&C\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\qquad(4.4)$$

for

$$\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}E\\ k\end{pmatrix},\begin{pmatrix}\Delta E\\ \Delta k\end{pmatrix},\begin{pmatrix}t\\ r\end{pmatrix},\begin{pmatrix}q_0\\ Q\end{pmatrix}$$

The variable $\zeta$ in the coefficients (4.2) is given in terms of the velocity $\beta$ of this boost as $\zeta=\mathrm{arctanh}\beta$.

$$\begin{aligned}C(q_0-E)-S(Q-k)&=q_0'-E'\\ E^2-k^2&={E'}^2-{k'}^2\\ Et-kr&=E't'-k'r'\end{aligned}\qquad(4.5)$$

#### 4.1.1 Harmonic analysis

In the sharp energy system (ses) the wave function for an unstable particle with energy $E>m$ is

$$\begin{aligned}\psi(t,r)&\overset{\mathrm{ses}}{=}-\int\frac{d^4q}{(2\pi)^2}\frac{\delta(q_0-E)}{q^2-m^2+im\Gamma}\mathrm{e}^{-xiq}\\ &=\mathrm{e}^{-iEt}\frac{\mathrm{e}^{i\underline{k}r}}{r}\end{aligned}\qquad(4.6)$$

with $\Gamma > 0$ and

$$\underline{k} = \sqrt{k^2 + im\Gamma}, \text{ where } k^2 = E^2 - m^2. \tag{4.7}$$

The case of the general system requires the new Dirac delta function (4.1) in the Fourier integral. The integral is first solved in the sharp energy system (primed variables), the result (4.8) is then Lorentz-transformed into the general system (unprimed variables).

$$\begin{aligned}\psi(t,r) &= -\int \frac{d^4q}{(2\pi)^2} \frac{\delta(C(q_0 - E) - S(Q - k))}{q^2 - m^2 + im\Gamma} \mathrm{e}^{-xiq} \\ &= \mathrm{e}^{-iE't'} \frac{\mathrm{e}^{i\underline{k}'r'}}{r} \\ &= \mathrm{e}^{-iEt} \frac{\mathrm{e}^{ikr}}{r} \mathrm{e}^{i(\underline{k}' - k')r'}\end{aligned} \tag{4.8}$$

because of (4.5). Here

$$\underline{k}' - k' = \sqrt{k'^2 + im\Gamma} - k' \text{ which becomes } \frac{im\Gamma/2}{k'} \tag{4.9}$$

to first order of $m\Gamma/k'^2$. This dimensionless parameter is of the order $\Gamma/m \ll 1$. Inserting (4.4) we find the factor

$$(\underline{k}' - k')r' = \frac{im\Gamma/2}{-SE + Ck}(-St + Cr) \tag{4.10}$$

which, using (4.2) and (2.12), turns out to be $-i\Delta Et + i\Delta kr$. Therefore the relativistic wave function of the unstable state is

$$\boxed{\psi(t,r) = \mathrm{e}^{-i(E - i\Delta E)t} \; \frac{\mathrm{e}^{i(k - i\Delta k)r}}{r}} \tag{4.11}$$

to first order of $\Gamma/m$.

In the central rest frame (crs) the wave function takes the form

$$\psi(t,r) \overset{\text{crs}}{=} \mathrm{e}^{-i(m - i\Gamma/2)t} \frac{\mathrm{e}^{-|\Delta k^*|r}}{r} \tag{4.12}$$

Note that $\Delta k^* \leq -\Gamma/2 < 0$ in the central rest frame of the decaying state. An unstable state unavoidably has some spatial extension.

If one does not want to go to the first order approximation in $\Gamma/m$ one may define a complex constant $\eta$ in (4.9)

$$\underline{k}' - k' = \sqrt{k'^2 + im\Gamma} - k' \overset{\text{def}}{=} i\eta B \tag{4.13}$$

which is 1 in the first order of $\Gamma/m$. It may be expressed in terms of a ratio of invariants of (2.12) and is

$$i\eta = \frac{m\Gamma/2}{B^2}\left[1 - \sqrt{1 + i\frac{4B^2}{m\Gamma}}\right] \tag{4.14}$$

With it the wave function generally takes the form

$$\psi(t,r) = \mathrm{e}^{-i(E - i\eta\Delta E)t} \; \frac{\mathrm{e}^{i(k - i\eta\Delta k)r}}{r} \tag{4.15}$$

Looking back at the integral (4.8), we see how the $\delta$-function picks out the particular solution from the Lorentz-invariant reservoir of all possible solutions. We note that the variables $E, k, \Delta E, \Delta k$ and their invariants $m^2, m\Gamma$ and $B^2$ (c.f. (2.12)) fall into two categories. The first comprises $m^2$ and $m\Gamma$ – they belong to the Lorentz-invariant part of the integrand (4.8) and are therefore a universal description of the unstable state, independent of a particular Lorentz frame or any particular production conditions. The second comprises the others, $E, k, \Delta E, \Delta k$ and $B$ – they appear in the $\delta$-function of the integrand (4.8) and therefore describe the unstable state in the particular conditions in which it was produced. There is an analogy with the spin of stable particles: the magnitude of spin is a universal property of the particle (mathematically characterized by the dimension of the representation space), the spin component is a particular property of the particle which depends on the production conditions (mathematically characterized by an eigenvector in the representation space).

A suitable notation would separate the two categories of variables. The unstable state, characterized by 4 real numbers, could be written as $|m^2, m\Gamma; k, \Delta k\rangle$ or $|m^2, m\Gamma; E, \Delta E\rangle$, the universal variables to the left of the semikolon, the experiment-dependent to the right. The corresponding notation for a spin state (of a stable particle) would be $|S; S_3\rangle$.

### 4.1.2 Comparison of (4.11) with the traditional ansatz

Traditionally [2] [8] one describes a decaying state in its rest system with the ansatz

$$\psi(t,r) \overset{\text{rs}}{=} \frac{1}{r}\mathrm{e}^{-i(m - i\Gamma/2)t} \tag{4.16}$$

If the state has energy-momentum $(E, k)$, (4.16) is replaced by

$$\psi(t,r) = \mathrm{e}^{-i(E - i\Delta E)t} \; \frac{\mathrm{e}^{i(k - i\Delta k)r}}{r} \tag{4.17}$$

The imaginary parts traditionally also form an energy-momentum four-vector $(\Delta E, \Delta k)$ which is the result of the same Lorentz boost that transforms

$$\begin{aligned}(m, 0) &\longrightarrow (E, k) \\ \left(\frac{\Gamma}{2}, 0\right) &\longrightarrow (\Delta E, \Delta k) = \left(\frac{E}{m}\frac{\Gamma}{2}, \frac{k}{m}\frac{\Gamma}{2}\right)\end{aligned} \tag{4.18}$$

The difference in the two approaches is in the norm of $(\Delta E, \Delta k)$; our spread vector is spacelike, the traditional one is timelike. The traditional spread vector keeps the same ratio of the momentum over the energy components in all Lorentz frames:

$$\left.\frac{\Delta k}{\Delta E}\right|_{\text{traditional}} = \frac{k}{E} \tag{4.19}$$

Traditionally, there is no sharp energy system, and the momentum spread in the particle's rest frame is zero.

$$\int \frac{dq_z e^{-i(E+(q_z-k_z)\Delta E/\Delta k_z)t+i(k_x+(q_z-k_z)\Delta k_x/\Delta k_z)x+i(k_y+(q_z-k_z)\Delta k_y/\Delta k_z)y+iq_z z}}{i\pi D}$$

$$D = (q_z - k_z)^2 \frac{-B^2}{\Delta k_z^2} + (q_z - k_z)\frac{m\Gamma}{\Delta k_z} + im\Gamma \tag{4.26}$$

This difference apart, (4.17) and (4.11) are the same. In a practical measurement of a continuous spherical wave emanating from a target at r = 0 one would measure the presence of the decaying particle as a function of the time in the particle's rest frame, after eliminating the space argument by writing $r = t\beta = tk/E$. Both (4.11) and (4.17) then take the form

$$\begin{aligned} \psi(t) &= \frac{1}{r} e^{[-i(E-i\Delta E)+i(k-i\Delta k)k/E]t} \\ &= \frac{1}{r} e^{-i[(E^2-k^2)-iE\Delta E+ik\Delta k]t/E} \\ \psi(t) &= \frac{1}{r} e^{[-i(m-i\Gamma/2)m/E]t}; \\ \psi(t^*) &= \frac{1}{r} e^{-i(m-i\Gamma/2)t^*} \end{aligned} \tag{4.20}$$

using (2.12). Here $t^* = tm/E$ is the time in the particle rest frame. This shows that our approach produces the same result as the traditional approach does – the spacelike nature of the energy-momentum spread in the first approximation has no consequence for the wave function. Also the accurate form (4.15) does not lead to a qualitatively different behavior.

It is only when spin is included that the spacelike nature of $(\Delta E, \Delta k)$ becomes essential.

We would like to remark that (4.20) also show the role of $\Gamma$ as an inverse lifetime. This follows naturally from the definition of the wave function in (4.8). The equality of $\Gamma$ with the inverse lifetime of the unstable state is the essential hypothesis in the work of A. Bohm and collaborators [7] for the construction of a unique Gamow vector for relativistic unstable states.

### 4.1.3 Fully specified energy-momentum 4-vector and spread 4-vector

Now let $(E, \vec{k})$ and $(\Delta E, \vec{\Delta k})$ be completely specified (8 real numbers). The three invariants are in this case:

$$\begin{aligned} &E^2 - \vec{k}^2 = m^2 > 0 \\ &(\Delta E)^2 - (\vec{\Delta k})^2 = -B^2 < 0 \\ &E\Delta E - \vec{k} \cdot \vec{\Delta k} = m\Gamma/2 > 0 \end{aligned} \tag{4.21}$$

The wave function $\psi(t, \vec{x})$ of this state is defined by harmonic analysis such that in the Fourier integral over energy momentum $(q_0, \vec{q})$ the deviations from the central values $(E, \vec{k})$ satisfy the following relations

$$\frac{q_0 - E}{\Delta E} = \frac{q_x - k_x}{\Delta k_x} = \frac{q_y - k_y}{\Delta k_y} = \frac{q_z - k_z}{\Delta k_z} \tag{4.22}$$

This can be achieved in the Fourier integral by the product of three delta distributions $\delta_I \delta_{II} \delta_{III}$

$$\begin{aligned} \delta_I &= \delta[(\Delta k_z/B)(q_0 - E) - (\Delta E/B)(q_z - k_z)] \\ \delta_{II} &= \delta[(\Delta k_z/B)(q_x - k_x) - (\Delta k_x/B)(q_z - k_z)] \\ \delta_{III} &= \delta[(\Delta k_z/B)(q_y - k_y) - (\Delta k_y/B)(q_z - k_z)] \end{aligned} \tag{4.23}$$

Here we have assumed $\Delta k_z \neq 0$. (There is always one $\Delta k_i \neq 0$.) The normalizing factor $B$ appears for the reason mentioned after (4.2).

We define the relativistic wave function to be

$$\psi(t, \vec{x}) = \int d^4q \delta_I \delta_{II} \delta_{III} \frac{-i/\pi}{q^2 - m^2 + im\Gamma} e^{-iqx} \tag{4.24}$$

The argument of $\delta_I$ vanishes when $q_0 = E + (q_z - k_z) \times \Delta E/\Delta k_z$. After integrating over $q_0$ the remaining integral (up to constant factors) is

$$\int d^3q \frac{e^{-i(E+(q_z-k_z)\Delta E/\Delta k_z)t} e^{i\vec{q}\,\vec{x}} \delta_{II}\delta_{III}}{(E+(q_z-k_z)\Delta E/\Delta k_z)^2 - q_x^2 - q_y^2 - q_z^2 - m^2 + im\Gamma} \tag{4.25}$$

After the integrals over $q_x$ and $q_y$ one is left with (see (4.26) on top of the page). The relevant zero of the denominator $D$ ist at

$$\underline{q_z} = k_z - i\eta\Delta k_z; \tag{4.27}$$

here $i\eta$ is related to the ratio of invariants of (4.21) as given by (4.14). Inserting $\underline{q_z}$ into the integrand yields the wave function up to a constant factor which has to be provided by a proper normalisation. We have

$$\psi(t, \vec{x}) = e^{-i(E-i\eta\Delta E)t} e^{i(\vec{k} - i\eta\vec{\Delta k})\cdot\vec{x}} \tag{4.28}$$

In first order of $\Gamma/m$, $\eta$ is equal to 1. This yields

$$\boxed{\psi(t, \vec{x}) = e^{-i(E-i\Delta E)t} e^{i(\vec{k} - i\vec{\Delta k})\cdot\vec{x}}} \tag{4.29}$$

Expression (4.29) is the relativistic wave function to first order in $\Gamma/m$ for a plane wave propagating in the direction $\vec{k}$ with energy $E$ and energy-momentum spread $(\Delta E, \vec{\Delta k})$. $\vec{k}$ and $\vec{\Delta k}$ are not necessarily parallel. Between the 8 real parameters there are the three invariants of (4.21). In the notation proposed at the end of Sect. 4.1.1, such a state is characterized as $|m^2, m\Gamma; \vec{k}, \vec{\Delta k}\rangle$.

## 4.2 Non-orthogonality of unstable states

Now we have the relativistic wave functions in space and time of the unstable states, we would like to interpret them in terms of probabilities. In non-relativistic quantum mechanics, the magnitude squared is the probability density, and the scalar product of two wave functions is the space integral of the product of one with the complex conjugate of the other.

In a relativistic framework the time component must be included. A probability interpretation of the relativistic wave functions has to be formulated in a Lorentz compatible way. This task is not undertaken here; but for one special Lorentz frame, the sharp energy system (SES), we may already formulate the scalar product involving unstable states.

The SES of an unstable state is defined as the Lorentz frame in which $\Delta E = 0$ (cf Sec. 2.3). Consider a stable and an unstable state with different masses $(M, m)$ and momenta $(K, k)$ but the same energy in the SES:

$$\text{Stable state } |\psi_s(K,M)\rangle = \frac{1}{r}\mathrm{e}^{-iEt}\mathrm{e}^{iKr} \tag{4.30}$$

$$\text{Unstable state } |\psi_u(k,m,\kappa)\rangle = \frac{1}{r}\mathrm{e}^{-iEt}\mathrm{e}^{i(k+i\kappa)r} \tag{4.31}$$

The presence of two such states is quite common for reactions of type (1.2) at sufficiently high momenta where in the forward direction the SES coincides with the laboratory system (in which the target was initially at rest). The momenta are given by $K = \sqrt{E^2 - M^2}$ and $k = \sqrt{E^2 - m^2}$.

Let the stable state be normalised as usual by the Dirac delta distribution

$$\langle\psi_s(K_2,M)|\psi_s(K_1,M)\rangle = \delta(K_2 - K_1) \tag{4.32}$$

This is the same as writing

$$\psi_s(K_i,M) = \frac{1}{\pi\sqrt{2}}\frac{\sin K_i r}{r} \qquad (i = 1,2; \quad K_i > 0) \tag{4.33}$$

$$\begin{aligned}&\langle\psi_s(K_2,M)|\psi_s(K_1,M)\rangle\\ &= \int_0^\infty \psi_s(K_1,M)\psi_s^*(K_2,M)4\pi r^2\, dr\\ &= \delta(K_2 - K_1)\end{aligned} \tag{4.34}$$

The standing spherical wave $\sin(Kr)/r$ is appropriate for normalisation purposes. Equivalently one may have taken the outgoing spherical wave $\mathrm{e}^{iKr}/r$ plus a prescription for the integration path.

For the scalar product between the stable and the unstable state we now take a generalisation of (4.33) and (4.34):

$$\begin{aligned}&\psi_u(k_i,m,\kappa) = \frac{1}{\pi\sqrt{2}}\frac{\sin k_i r}{r}\mathrm{e}^{-\kappa r}\\ &\quad (i = 1,2; \quad k_i > 0)\end{aligned} \tag{4.35}$$

$$\begin{aligned}&\langle\psi_s(K_2,M)|\psi_u(k_1,m,\kappa)\rangle\\ &= \int_0^\infty \psi_u(k_1,m,\kappa)\psi_s^*(K_2,M)4\pi r^2\, dr\\ &= \frac{\kappa/\pi}{(k_1 - K_2)^2 + \kappa^2}\end{aligned} \tag{4.36}$$

This is the scalar product in the SES of two relativistic wave functions, one stable with real momentum $K$, the other unstable with complex momentum $k + i\kappa$. The imaginary part $\kappa > 0$ sets the scale on which the two momenta $k_1$ and $K_2$ are allowed to be different before the scalar product vanishes. Expression (4.36) is valid at momenta large compared to $\kappa$ and $k_1 - K_2$ because small second-order terms $\kappa^2/(k_1 + K_2)^2$ and $(k_1 - K_2)^2/(k_1 + K_2)^2$ were neglected.

Analogously the normalisation of the unstable state is given by

$$\begin{aligned}&\langle\psi_u(k_2,m,\kappa)|\psi_u(k_1,m,\kappa)\rangle\\ &= \int_0^\infty \psi_u(k_1,m,\kappa)\psi_u^*(k_2,m,\kappa)4\pi r^2\, dr\\ &= \frac{2\kappa/\pi}{(k_1 - k_2)^2 + (2\kappa)^2}\end{aligned} \tag{4.37}$$

This last expression is valid in first order of $\kappa/k$ and tends to the delta distribution $\delta(k_1 - k_2)$ as $\kappa$ goes to zero. In contrast, (4.36) vanishes with $\kappa$ because $k_1$ and $K_2$ cannot be equal in the SES as long as $m \neq M$.

## 5 Inclusion of spin

It is well known that two simultaneous eigenstates $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$ of the operators $\mathbf{J}$ and $\mathbf{J_z}$ of total angular momentum and angular momentum component are orthogonal on each other for different quantum numbers:

$$\langle j_1, m_1|j_2, m_2\rangle = 0 \text{ if } j_1 \neq j_2 \text{ or } m_1 \neq m_2 \tag{5.1}$$

We propose that (5.1) has to be modified if one of the two states is short-lived.

For this we first mention three arguments why the angular momentum of short-lived states cannot be infinitely sharp but acquires an uncertainty which we call 'angular momentum spread' (Sect. 5.1). Since angular momentum eigenvalues are discrete numbers (integer or half integer) as a consequence of the compact nature of the underlying rotation group, an uncertainty in angular momentum, or spin spread can only mean the occupation of neighbouring states in addition to the original main state. Insofar as the spin spread is a small effect one may think of a perturbation so that the original main spin retains its well defined meaning.

We then have to develop a formalism for the description of spin for relativistic unstable particles which is open to the possibility of non-orthogonal scalar products between states of different quantum numbers. With this in mind, we review for stable particles in Sects. 5.2 and 5.3 how the spin group is embedded in the Lorentz group and how relativistic wave functions for a given spin state can be constructed.

On this basis we proceed in Sect. 5.4 to a generalization valid for unstable states. The new element is the momentum spread in the particle's central rest frame, it adds a spin-1 property to the original main spin, creating 'spin neighbours' around the original spin. Therefore some room must be created plus and minus one unit of spin in the representation of the spin group for a given particle. This is achieved by designating a higher representation of the Lorentz group than done for the stable case.

## 5.1 Three direct arguments in favour of the angular momentum spread of short-lived particles

### 5.1.1 The spin of a short-lived state can only be measured with a limited accuracy

Imagine an experiment in which the amount $J$ of spin angular momentum of a short-lived state is measured using the method of nuclear spin resonance in a magnetic field $B$. The precession frequency $\omega$ which is observed for non zero spin determines $J$, if the magnetic moment $\mu$ is known; then

$$J = \frac{\mu B}{\omega}.$$

Let the excited state have energy $E$, above the ground level $E_0$, width $\Gamma$ and mean life $\tau = \hbar/\Gamma$. Now the frequency can only be measured during the lifetime of the state and is therefore limited to the accuracy $\Delta\omega \approx 1/\tau$, so that the accuracy of the measurement of $J$ is given by

$$\Delta J \approx \frac{\Delta\omega}{\omega^2}\mu B \approx \frac{1}{\tau}\frac{J^2}{\mu B},$$

which becomes smallest for the largest value of $\mu B$, and a small $J$.

In this experiment the interaction energy between the apparatus and the excited state is represented by $\mu B$; it cannot be made as large as the value of $E - E_0$ if the state to be measured should remain distinguished from the ground state. Therefore we have to keep $\mu B < E - E_0$. Assigning the lowest integer non-zero value to $J$, $J = \hbar$, we arrive at an uncertainty that cannot be smaller than

$$\Delta J \approx \frac{1}{\tau}\frac{\hbar^2}{E - E_0} = \frac{\hbar\Gamma}{E - E_0}$$
$$\frac{\Delta J}{\hbar} \approx \frac{\Gamma}{E - E_0} \qquad (5.2)$$

### 5.1.2 The angular momentum component of a short-lived state can only be measured with a limited accuracy

In a Stern-Gerlach experiment let a beam of short-lived states of energy $E$ be created from a beam of atoms that are in the ground state with energy $E_0$. In the magnetic field $B$ the presence of an angular momentum component $m\hbar$ in the direction of the field inhomogeneity changes the energy to be $E' = E - \mu B$, where $\mu$ is the magnetic moment of the state. Knowing $\mu$, one measures $m$ by recording the energy difference; then $m$ is given by

$$m = \frac{E' - E}{\mu B}.$$

The energy difference $E' - E$ is determined from a measurement of the displacement of the beam in the inhomogeneous field.

It is well known that a minimal length of time, $T$, is required to measure the energy difference $E' - E$; for shorter times the beam would still overlap the reference beam.

$$T > \frac{\hbar}{E' - E}.$$

The unstable state can only be observed during its lifetime; on the average we must involve the mean life time $\tau$. Therefore there is a limit to the accuracy $\Delta(E - E')$ with which the value of $E' - E$ can be determined:

$$\Delta(E' - E) > \frac{\hbar}{\tau}$$
$$\Delta m = \frac{\Delta(E - E')}{\mu B} > \frac{\hbar}{\tau\mu B}.$$

The accuracy $\Delta m$ becomes better if $\mu B$ is increased. But again, $B$ cannot be made arbitrarily large. If $\mu B$ were made as large as the energy difference $E - E_0$, the experiment would lose its meaning as the state $E$ would no longer be distinguished from $E_0$. Therefore, the uncertainty $\Delta m$ with which the angular momentum component can be measured has a lower limit

$$\Delta m > \frac{\hbar}{\tau(E - E_0)} = \frac{\Gamma}{E - E_0}, \qquad (5.3)$$

where $\Gamma = \hbar/\tau$ is the width of the unstable state.

### 5.1.3 A short-lived state cannot have a sharp orbital angular momentum

In a two body bound state the angle $\varphi$ of rotational motion, and the angular momentum $L$ of the state are two variables that are conjugate to each other, and the uncertainty principle requires that the two uncertainties $\Delta\varphi$ and $\Delta L$ are related by $\Delta\varphi\, \Delta L > \hbar$. Whereas for a stable state the uncertainty $\Delta\varphi$ is arbitrarily large, this is different for a short-lived state because $\varphi(t)$, being essentially proportional to the time $t$, must be limited in the same way as the lifetime is. A state with a finite mean life $\tau$ has its lifetime distributed according to a probability distribution $(1/\tau)\exp(-t/\tau)$. The variance of the lifetime is therefore equal to

$$(\Delta t)^2 = [t^2] - [t]^2 = \tau^2 = \frac{\hbar^2}{\Gamma^2}.$$

If we use $\alpha_{\text{eff}}$ as the effective constant of proportionality, so that

$$\varphi(t) \approx \alpha_{\text{eff}} t$$

then

$$\Delta\varphi \approx \alpha_{\text{eff}}\tau$$

and $\Delta L$ is bounded from below at approximately

$$\Delta L \approx \frac{\hbar}{\alpha_{\text{eff}}\tau}.$$

The physical meaning of $\alpha_{\text{eff}}$ is the angular velocity, which in the classical problem is related to the angular momentum and the kinetic energy by

$$L \approx M\alpha_{\text{eff}} r^2$$
$$E_{kin} \approx \alpha_{\text{eff}}^2 r^2 M/2$$

where $r$ is the suitably averaged radius und $M$ is the mass of the two body problem. Therefore $\alpha_{\text{eff}}$ is of the order of $E_{kin}/L$, and the lower bound for $\Delta L$ can also be written as

$$\Delta L \approx \frac{\hbar}{\tau}\frac{L}{2E_{kin}}$$
$$\frac{\Delta L}{L} \approx \frac{\Gamma}{2E_{kin}} \tag{5.4}$$

## 5.2 Lorentz group embedding of the spin group

For relativistically compatible wave functions, spinning massive particles have to be embedded into relativistic particle fields with Lorentz group representations. The mathematical method employed is the induction of Lorentz group representations by spin group representations [3,4].

A finite dimensional Lorentz group $\mathbf{SL}(\mathbb{C}^2)$ representation is decomposable into spin subgroup $\mathbf{SU}(2)$ representations

$$\mathbf{SL}(\mathbb{C}^2) \cong \bigoplus \mathbf{SU}(2):\ D^{[L|R]} \cong \bigoplus_{S=|L-R|}^{L+R} D^S \tag{5.5}$$
$$\text{e.g. } D^{[0|0]} \cong D^0,\ D^{[J|0]} \cong D^J,\ D^{[\frac{1}{2}|\frac{1}{2}]} \cong D^1 \oplus D^0$$
$$D^{[1|1]} \cong D^2 \oplus D^1 \oplus D^0$$

In contrast to the spin decompositions of Lorentz group representations, the embedding of spin $S$ particles into a Lorentz transformation compatible field is not unique since a given spin $S$-representation can be found in many Lorentz group representations. Mathematically speaking, the induced Lorentz group representation is highly reducible [3,4]. In order to make the embedding unique one may use the additional rule, applied to stable particles, that the 'left and right spin' indices $L$ and $R$ should be chosen as small as possible and as close as possible to each other.

$$S = L + R \text{ and } |L-R| = \begin{cases} 0 & \text{for spin } S = 0, 1, \ldots \\ \frac{1}{2} & \text{for spin } S = \frac{1}{2}, \frac{3}{2}, \ldots \end{cases} \tag{5.6}$$

This prescription gives equal 'left-right spin' for the relativistic embedding of integer spin $S$ and a 'left-right spin' difference $\frac{1}{2}$ for the embedding of halfinteger spin

$$\mathbf{SU}(2) \hookrightarrow \mathbf{SL}(\mathbb{C}^2) \cong \bigoplus \mathbf{SU}(2):$$
$$D^S \hookrightarrow \begin{cases} D^{[\frac{S}{2}|\frac{S}{2}]} \cong \bigoplus_{J=0}^{S} D^J \\ \quad \text{for } S = 0, 1, \ldots \\ D^{[\frac{S+\frac{1}{2}}{2}|\frac{S-\frac{1}{2}}{2}]} \oplus D^{[\frac{S-\frac{1}{2}}{2}|\frac{S+\frac{1}{2}}{2}]} \cong 2 \times \bigoplus_{J=\frac{1}{2}}^{S} D^J \\ \quad \text{for } S = \frac{1}{2}, \frac{3}{2} \ldots \end{cases} \tag{5.7}$$

Two examples for integer spin, e.g. for scalar and vector particles like stable $\pi$ and $\rho$, are

$$D^0 \hookrightarrow D^{[0|0]} \cong D^0,\ D^1 \hookrightarrow D^{[\frac{1}{2}|\frac{1}{2}]} \cong D^1 \oplus D^0 \tag{5.8}$$

To make the embedding unique for halfinteger spin, the selfconjugated sum of two conjugated Weyl representations (left and right) is used, as familiar from the Dirac field for the spin $\frac{1}{2}$ electron with the Dirac representation

$$D^{\frac{1}{2}} \hookrightarrow D^{[\frac{1}{2}|0]} \oplus D^{[0|\frac{1}{2}]} \cong 2 \times D^{\frac{1}{2}} \tag{5.9}$$

Such a doubling leads to finite dimensional unitary representations of the Lorentz group - of course, indefinite unitary, e.g. in $\mathbf{SU}(2,2)$ for Dirac spinors.

The Lorentz transformation from a rest system for a particle with mass $m > 0$ to a general system is effected by the two Weyl representations $s$, $\hat{s}$ of the boosts

$$s\left(\frac{q}{m}\right) = \mathrm{e}^{\frac{\vec{\beta}\vec{\sigma}}{2}},\ \hat{s}\left(\frac{q}{m}\right) = \mathrm{e}^{-\frac{\vec{\beta}\vec{\sigma}}{2}}$$
$$\text{with } \vec{\beta} = \frac{\vec{q}}{|\vec{q}|}\text{artanh}\frac{|\vec{q}|}{q_0}$$
$$s\left(\frac{q}{m}\right) = \sqrt{\frac{q_0+m}{2m}}[\mathbf{1}_2 + \frac{\vec{\sigma}\vec{q}}{q_0+m}]$$
$$= \frac{1}{\sqrt{2m(q_0+m)}}\begin{pmatrix} q_0+m+q^3 & q^1-iq^2 \\ q^1+iq^2 & q_0+m-q^3 \end{pmatrix}$$
$$\hat{s}\left(\frac{q}{m}\right) = \sqrt{\frac{q_0+m}{2m}}[\mathbf{1}_2 - \frac{\vec{\sigma}\vec{q}}{q_0+m}]$$
$$= \frac{1}{\sqrt{2m(q_0+m)}}\begin{pmatrix} q_0+m-q^3 & -q^1+iq^2 \\ -q^1-iq^2 & q_0+m+q^3 \end{pmatrix}$$
$$s\left(\frac{q}{m}\right) \overset{\text{rs}}{=} \mathbf{1}_2 = s(1,0,0,0),\ \hat{s}\left(\frac{q}{m}\right)$$
$$\overset{\text{rs}}{=} \mathbf{1}_2 = \hat{s}(1,0,0,0) \tag{5.10}$$

All transmutators from spin $\mathbf{SU}(2)$-representations to embedding finite dimensional Lorentz group $\mathbf{SL}(\mathbb{C}^2)$-representations can be built via totally symmetrized products of those two fundamental spin-Lorentz transmutators in the Weyl representations

$$D^{[L|R]}\left(\tfrac{q}{m}\right) = \bigvee^{2L} s\left(\tfrac{q}{m}\right) \otimes \bigvee^{2R} \hat{s}\left(\tfrac{q}{m}\right) \tag{5.11}$$

They are representations of the boosts as homogeneous space

$$\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathbf{SO}_0(1,3)/\mathbf{SO}(3) \qquad (5.12)$$

For vector representations $D^{[\frac{1}{2}|\frac{1}{2}]}$ – e.g. acting on the energy-momentum space itself – the spin-Lorentz transmutators are the vector representations of the boosts

$$\begin{aligned} D^{[\frac{1}{2}|\frac{1}{2}]}(\frac{q}{m}) &= \Lambda(\frac{q}{m})^i_k \cong \frac{1}{2}\operatorname{tr} s(\frac{q}{m})\sigma^i s^\star(\frac{q}{m})\check{\sigma}_k = \mathrm{e}^{\begin{pmatrix} 0 & \vec{\beta} \\ \vec{\beta} & 0 \end{pmatrix}} \\ &= \frac{1}{m}\begin{pmatrix} q_0 & \vec{q} \\ \vec{q} & \check{\mathrm{e}}^{ab}m + \frac{q^a q^b}{q_0+m} \end{pmatrix} \\ \Lambda(\frac{q}{m}) &\overset{\text{rs}}{=} \Lambda(1,0,0,0) = \mathbf{1}_4,\ \Lambda(\frac{q}{m})\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \\ &= \frac{1}{m}\begin{pmatrix} q_0 \\ \vec{q} \end{pmatrix} \end{aligned} \qquad (5.13)$$

with the Weyl matrices $\sigma^i = (\mathbf{1}_2, \vec{\sigma}) = \check{\sigma}_j$. The four columns of the matrix $\Lambda(\frac{q}{m})^i_{0,a} (a, b = 1, 2, 3)$ are the components of a general basis in energy-momentum space, for $\vec{q} = 0$ a rest system. Therefore we have the $\mathbf{SO}_0(1,3)$-orthogonality and $\mathbf{SO}(3)$-projector conditions

$$\begin{aligned} &\Lambda(\tfrac{q}{m})^i_{0,a}\eta_{ij}\Lambda(\tfrac{q}{m})^j_{0,b} = \begin{pmatrix} 1 & 0 \\ 0 & -\delta_{ab} \end{pmatrix} \\ &\Lambda(\tfrac{q}{m})^i_0 = \tfrac{q^i}{m},\ \Lambda(\tfrac{q}{m})^i_a \check{\mathrm{e}}^{ab}\Lambda(\tfrac{q}{m})^j_b = -\eta^{ij} + \tfrac{q^i q^j}{m^2} \end{aligned} \qquad (5.14)$$

## 5.3 Relativistic wave functions for spin 1 stable particles

As an example we calculate the wave function for a stable massive spin-1 particle. According to our general method, we will induce [5] the unitary Poincaré group $\mathbf{SL}(\mathbb{C}^2)\vec{\times}\mathbb{R}^4$ representations by the spin-time $\mathbf{SU}(2)\times\mathbb{R}$ representation. For this purpose we combine the boost representation (the spin-Lorentz transmutators) of the foregoing subsection with the embedded translations described above as $\delta(E^2 - m^2) \hookrightarrow \delta(q^2 - m^2)$.

For a spin-1 particle, $\mathbf{SO}(3)$-vectors ($\mathbf{SU}(2)$ representation with spin $S = 1$) are embedded with a $[\frac{1}{2}|\frac{1}{2}]$-spin-Lorentz transmutator

$$\begin{aligned} \mathbf{Z}^i(x) &= \int \frac{d^3q}{(2\pi)^3}\Lambda(\frac{q}{m})^i_a \frac{\mathrm{e}^{-iqx}\mathrm{u}^a(\vec{q}) + \mathrm{e}^{iqx}\mathrm{u}^{\star a}(\vec{q})}{\sqrt{2q_0}} \\ &\text{with}\quad q = (q_0, \vec{q}),\ q_0 = \sqrt{m^2 + \vec{q}^2}, \\ &\qquad i = 0,1,2,3,\ a = 1,2,3 \end{aligned} \qquad (5.15)$$

The momentum $\vec{q} \in \mathbb{R}^3$-indexed creation and annihilation operators are eigenvectors with respect to both spacetime translations and spin rotations.

The on-shell quantization commutators and anticommutator Fock forms

$$\begin{pmatrix} [\mathbf{Z}^i, \mathbf{Z}^j](x) \\ \langle\{\mathbf{Z}^i, \mathbf{Z}^j\}\rangle_{\mathrm{F}}(x) \end{pmatrix} = \int \frac{d^4q}{(2\pi)^3}\begin{pmatrix} \epsilon(q_0) \\ 1 \end{pmatrix} \times \left(-\eta^{ij} + \frac{q^i q^j}{m^2}\right)\delta(q^2 - m^2)\mathrm{e}^{-iqx} \qquad (5.16)$$

are used in the Feynman propagator which displays the embedded matrix $\mathbf{1}_3 \cong \check{\mathrm{e}}^{ab}$ of a triplet $\mathbf{SO}(3)$-representation

$$\begin{aligned} &\langle\{\mathbf{Z}^i, \mathbf{Z}^j\}(x) - \epsilon(x_0)[\mathbf{Z}^i, \mathbf{Z}^j](x)\rangle_{\mathrm{F}} \\ &= \tfrac{i}{\pi}\int \tfrac{d^4q}{(2\pi)^3}\tfrac{-\eta^{ij} + \frac{q^i q^j}{m^2}}{q^2 + io - m^2}\mathrm{e}^{-iqx} \\ &= \tfrac{i}{\pi}\int \tfrac{d^4q}{(2\pi)^3}\Lambda(\tfrac{q}{m})^i_a \tfrac{\check{\mathrm{e}}^{ab}}{q^2 + io - m^2}\Lambda(\tfrac{q}{m})^j_b \mathrm{e}^{-iqx} \end{aligned} \qquad (5.17)$$

The wave function for a stable spin 1 particle with energy $E$ - i.e. in any Lorentz system, is given by the 'square root' of the the Feynman propagator

$$\begin{aligned} Z^i_a(t,r) &= -\int \frac{d^4q}{2\pi^2}\Lambda\left(\frac{q}{m}\right)^i_a \frac{1}{q^2 + io - m^2}\mathrm{e}^{-iqx}\delta(q_0 - E) \\ &= -\Lambda\left(\frac{i\partial_t}{m}, \frac{-i\vec{\partial}}{m}\right)^i_a \int \frac{d^4q}{2\pi^2}\frac{1}{q^2 + io - m^2}\mathrm{e}^{-iqx}\delta(q_0 - E) \\ &= \mathrm{e}^{-iEt}\Lambda\left(\frac{E}{m}, \frac{-i\vec{\partial}}{m}\right)^i_a kh^+_0(kr) \end{aligned} \qquad (5.18)$$

The $\vec{q}$-dependence in the transmutator gives rise to a derivative with respect to the position of the $L = 0$ Hankel wave. It can be written as a radial derivative multiplied with the direction on the 2-sphere (spherical harmonics $\mathrm{Y}^1 \sim \frac{\vec{x}}{r}$)

$$\begin{aligned} \vec{\partial} &= \frac{\vec{x}}{r}d_r,\quad \frac{\vec{x}}{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}, \\ \frac{1}{r}\begin{pmatrix} x \pm iy \\ z \end{pmatrix} &= \begin{pmatrix} \sin\theta\mathrm{e}^{\pm i\varphi} \\ \cos\theta \end{pmatrix} \end{aligned} \qquad (5.19)$$

In general, the 'traceless' $\frac{\vec{x}}{r}$-monomials of degree $L$ in a spherical basis give the spherical harmonics $\mathrm{Y}^L_m$, e.g. for $L = 1, 2$

$$\begin{aligned} \mathrm{Y}^1_\bullet(\theta,\varphi) &\sim \frac{\vec{x}}{r}, \\ \mathrm{Y}^2_\bullet(\theta,\varphi) &\sim [\frac{\vec{x}}{r} \vee \frac{\vec{x}}{r}]_0 \sim \frac{x^a x^b - \frac{1}{3}\check{\mathrm{e}}^{ab}r^2}{r^2} \\ \begin{pmatrix} \mathrm{Y}^1_{\pm 1} \\ \mathrm{Y}^1_0 \end{pmatrix}(\theta,\varphi) &= \sqrt{\frac{3}{4\pi}}\begin{pmatrix} \mp\frac{1}{\sqrt{2}}\sin\theta\ \mathrm{e}^{\pm i\varphi} \\ \cos\theta \end{pmatrix}, \\ \begin{pmatrix} \mathrm{Y}^2_{\pm 2} \\ \mathrm{Y}^2_{\pm 1} \\ \mathrm{Y}^2_0 \end{pmatrix}(\theta,\varphi) &= \sqrt{\frac{5}{4\pi}}\begin{pmatrix} \sqrt{\frac{3}{8}}\sin^2\theta\ \mathrm{e}^{\pm 2i\varphi} \\ \mp\sqrt{\frac{3}{2}}\sin\theta\cos\theta\ \mathrm{e}^{\pm i\varphi} \\ \frac{3\cos^2\theta - 1}{2} \end{pmatrix} \end{aligned} \qquad (5.20)$$

The vector transmutator contains maximally two radial derivatives

$$\begin{aligned} \Lambda\left(\frac{E}{m}, \frac{-i\vec{\partial}}{m}\right) &= \frac{1}{m}\begin{pmatrix} E & -i\vec{\partial} \\ -i\vec{\partial} & \delta^{ab}m - \frac{\partial^a\partial^b}{E+m} \end{pmatrix} \\ &= \frac{1}{m}\begin{pmatrix} E & -\frac{\vec{x}}{r}id_r \\ -\frac{\vec{x}}{r}id_r & \delta^{ab}m - \frac{x^a x^b}{r^2(E+m)}d_r^2 \end{pmatrix} \end{aligned} \qquad (5.21)$$

When applied to the Hankel functions $h_0^+(kr)$, higher order Hankel functions [6] are produced up to angular momentum $l = 2$:

$$d_r h_0^+(kr) = -kh_1^+(kr),\ d_r^2 h_0^+(kr) = k^2\frac{2h_2^+(kr) - h_0^+(kr)}{3} \tag{5.22}$$

here we have used the derivative properties of spherical Bessel and Hankel functions (all called $f_l(\rho)$)

$$\begin{aligned} f_l(\rho) &= \rho^l \left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^l f_0(\rho) \\ &= -\rho^{l-1}\frac{d}{d\rho}\frac{f_{l-1}(\rho)}{\rho^{l-1}} \\ (2l+1)f_l(\rho) &= \rho[f_{l+1}(\rho) + f_{l-1}(\rho)] \end{aligned} \tag{5.23}$$

The intrinsic spin 1 of the particle combines with the $l = 1$ angular momentum $\frac{\vec{x}}{r}$ resulting in total spin $j = 0, 1, 2$.

Therewith, the three energy picked relativistically compatible wave functions $Z^i_{a=1,2,3}$ for a stable spin 1 massive particle are given by the three 4-vectors which are on the right hand side in the $(4 \times 4)$-matrix

$$\Lambda\left(\frac{E}{m}, -\frac{i\vec{\partial}}{m}\right) kh_0^+(kr) = \tag{5.24}$$

$$\left(\begin{array}{c||c} \frac{E}{m}\ kh_0^+(kr) & \frac{\vec{x}}{r}\ i\frac{k}{m}\ kh_1^+(kr) \\ \hline \frac{\vec{x}}{r}\ i\frac{k}{m}\ kh_1^+(kr) & \delta^{ab} kh_0^+(kr) - \frac{x^a x^b}{r^2}\ \frac{k^2}{m(E+m)}\ k\frac{2h_2^+(kr)-h_0^+(kr)}{3} \end{array}\right),$$

multiplied with $\mathrm{e}^{-iEt}$. The $a = 1$ component is explicitly

$$Z_1^{i=0,1,2,3}(t, r) = \tag{5.25}$$

$$\mathrm{e}^{-iEt}\left(\begin{array}{r} \frac{x}{r}\quad i\frac{k}{m}\quad kh_1^+(kr) \\ \hline kh_0^+(kr) - \frac{x^2}{r^2}\ \frac{k^2}{m(E+m)}\ k\frac{2h_2^+(kr)-h_0^+(kr)}{3} \\ -\frac{xy}{r^2}\ \frac{k^2}{m(E+m)}\ k\frac{2h_2^+(kr)-h_0^+(kr)}{3} \\ -\frac{xz}{r^2}\ \frac{k^2}{m(E+m)}\ k\frac{2h_2^+(kr)-h_0^+(kr)}{3} \end{array}\right)$$

with $k = \sqrt{E^2 - m^2}$. In a spherical basis the $\frac{x^a}{r}$-dependence can be expressed with the spherical harmonics above. When going to the rest system, the $a = 1$ component becomes

$$Z_1^{i=0,1,2,3}(t, r) \overset{\text{rs}}{=} \mathrm{e}^{-imt}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\frac{1}{r} + \cdots \tag{5.26}$$

The dots behind the last expression stand for the terms of higher order in $1/r$. They vanish as $r \gg 1/m$. For the spin components in the rest system we are interested in the factor in front of the $1/r$ term which is the one that survives at large $r$.

## 5.4 Unstable particles – spin neighbors

The spin definition of an unstable particle must take into account the additional spacelike spread vector.

As was discussed above for stable particles, the minimal embedding Lorentz group representation for a given $\mathbf{SU}(2)$-spin $S$ includes all spins up to $S$ – i.e. $\{0, \ldots, S\}$ for integer $S$ and $\{\frac{1}{2}, \ldots S\}$ for halfinteger $S$. To take into account the possible $S+1$-structure for unstable particles we choose for a minimal embedding a Lorentz group representation with the 'left and right spin' $L$ and $R$ indices increased, but as 'close as possible' to each other, so that we leave room for a *spin spread* $\Delta S = 1$. In comparison with the stable particle case, $S$ has to be replaced everywhere by $S+1$

$$\mathbf{SU}(2) \hookrightarrow \mathbf{SL}(\mathbb{C}^2) \cong \bigoplus \mathbf{SU}(2):$$

$$D^S \otimes D^1 \hookrightarrow \begin{cases} D^{[\frac{S+1}{2}|\frac{S+1}{2}]} \cong \bigoplus_{J=0}^{S+1} D^J, \\ \quad S = 0, 1, \ldots \\ D^{[\frac{S+\frac{3}{2}}{2}|\frac{S+\frac{1}{2}}{2}]} \oplus D^{[\frac{S+\frac{1}{2}}{2}|\frac{S+\frac{3}{2}}{2}]} \cong 2 \times \bigoplus_{J=\frac{1}{2}}^{S+1} D^J, \\ \quad S = \frac{1}{2}, \frac{3}{2} \ldots \end{cases} \tag{5.27}$$

with the examples

$$\begin{aligned} &S = 0: D^{[\frac{1}{2}|\frac{1}{2}]} \cong D^1 \oplus D^0,\ S = 1:\ D^{[1|1]} \cong D^2 \oplus D^1 \oplus D^0 \\ &S = \tfrac{1}{2}: D^{[1|\frac{1}{2}]} \oplus D^{[\frac{1}{2}|1]} \cong 2 \times [D^{\frac{3}{2}} \oplus D^{\frac{1}{2}}] \end{aligned} \tag{5.28}$$

### 5.4.1 Unstable particles with central spin 0

As mentioned above, the embedding of a spin representation into an induced Lorentz group representation is not unique. A stable spinless particle can be embedded into a $D^{[0|0]}$ field $\mathbf{A}$ as well as into its 1st derivative $\partial^j \mathbf{A}$ acted upon with a $D^{[\frac{1}{2}|\frac{1}{2}]}$ Lorentz group representation

$$\begin{aligned} &\partial^j \mathbf{A}(x) = -i\int \frac{d^3q}{q_0(2\pi)^3}\ \Lambda(\frac{q}{m})^j_0 \frac{\mathrm{e}^{-iqx}\mathrm{u}(\vec{q}) - \mathrm{e}^{iqx}\mathrm{u}^\star(\vec{q})}{\sqrt{2}}, \\ &\Lambda(\frac{q}{m})^j_0 = \frac{q^j}{m} \\ &\text{with}\quad q = (q_0, \vec{q}),\ q_0 = \sqrt{m^2 + \vec{q}^2} \end{aligned} \tag{5.29}$$

It involves for the boost transformation the spin-Lorentz transmutator $\Lambda(\frac{q}{m})^j_k$. The picked wave function for a stable spin 0 particle is

$$\begin{aligned} A^j(t, r) &= -\int \frac{d^4q}{2\pi^2}\Lambda\left(\frac{q}{m}\right)^j_0 \frac{1}{q^2 + io - m^2}\mathrm{e}^{-iqx}\delta(q_0 - E) \\ &= \mathrm{e}^{-iEt}\Lambda\left(\frac{E}{m}, -i\frac{\vec{\partial}}{m}\right)^j_0 \frac{\mathrm{e}^{ikr}}{r} \\ &= \mathrm{e}^{-iEt}\left(\begin{array}{c} \frac{E}{m} \\ \hline -\frac{\vec{x}}{r}\frac{i}{m}d_r \end{array}\right) kh_0^+(kr) \\ &= \mathrm{e}^{-iEt}\left(\begin{array}{c} \frac{E}{m}kh_0^+(kr) \\ \hline \frac{\vec{x}}{r}\ i\frac{k}{m}\ kh_1^+(kr) \end{array}\right) \end{aligned} \tag{5.30}$$

In the rest system this is

$$A^j(t,r) \stackrel{\text{rs}}{=} \mathrm{e}^{-imt} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{1}{r} + \cdots \tag{5.31}$$

(For the neglected terms, refer to the remarks following (5.26).) There is no new information compared with the wave function considered above that had only one scalar component $A(t,r) = \mathrm{e}^{-iEt}\, kh_0^+(kr)$. The wave function is just distributed to a Lorentz vector.

Things change if this method is applied to an unstable particle. Here the energy-momentum spread vector produces a contribution in the new components, thus exhibiting the spin spread for unstable particles.

As the simplest example, the wave function for a spinless unstable particle will be calculated.

$$\begin{aligned} A^j_\Gamma(t,r) &= -\int \tfrac{d^4q}{(2\pi)^2} \Lambda(\tfrac{q}{m})^j_0 \tfrac{1}{q^2 - m(m-i\Gamma)} \\ &\quad \times \mathrm{e}^{-iqx} \delta\Big(C(q_0 - E) - S(q_3 - k)\Big) \\ &= \begin{pmatrix} \frac{i\partial_t}{m} \\ -\frac{\vec{x}}{r}\frac{i}{m} d_r \end{pmatrix} \mathrm{e}^{-i(E - i\eta\Delta E)t} \tfrac{1}{r} \mathrm{e}^{i(k - i\eta\Delta k)r} \\ &= \mathrm{e}^{-i(E-i\eta\Delta E)t} \\ &\quad \times \begin{pmatrix} \frac{E - i\eta\Delta E}{m} kh_0^+(kr) \\ \frac{\vec{x}}{r} i[\frac{k}{m}\, kh_1^+(kr) - \frac{\eta\Delta k}{m}\, kh_0^+(kr)] \end{pmatrix} \mathrm{e}^{\eta\Delta k r} \end{aligned} \tag{5.32}$$

Genuine spin 1 contributions are proportional to the momentum spread as seen clearly in a central rest system $(E,k) \stackrel{\text{crs}}{=} (m,0)$

$$\begin{aligned} A^j_\Gamma(t,r) &\stackrel{\text{crs}}{=} \mathrm{e}^{-i(m - i\eta\frac{\Gamma}{2})t} \begin{pmatrix} 1 - \frac{i\eta\Gamma}{2m} \\ -\frac{\vec{x}}{r} i\eta\sqrt{\frac{B^2}{m^2} + \frac{\Gamma^2}{4m^2}} \end{pmatrix} \\ &\quad \times \frac{1}{r} \mathrm{e}^{-\eta\sqrt{B^2 + \frac{\Gamma^2}{4}}\, r} + \cdots \end{aligned} \tag{5.33}$$

The terms of order $1/r^2$ neglected in (5.33) correspond to the ones in (5.26) and (5.31). For the unstable state we cannot just go with $r$ to infinity but have to stay in the region where it has not yet decayed, say, $r \ll 1/\Delta k$. This requirement and the one that $r \gg 1/m$ can be simultaneously satisfied if $\Gamma/m \ll 1$. Under these conditions, and in leading order of $\Gamma/m$, the wave function for an unstable spin zero particle can be expressed with the momentum spread $\Delta k^*$ in the central rest frame (4.12) as

$$\boxed{A^j_\Gamma(t,r) \stackrel{\text{crs}}{=} \begin{pmatrix} 1 \\ -\frac{\vec{x}}{r}\frac{i\Delta k^*}{m} \end{pmatrix} \mathrm{e}^{-i(m - i\Gamma/2)t} \frac{\mathrm{e}^{-|\Delta k^*| r}}{r}} \tag{5.34}$$

The spin-1 neighbors (lower three components) are seen to arise proportionally to the momentum spread $\Delta k^*$ in the central rest frame of the particle. They are an immediate consequence of the short lifetime. The sum of the magnitudes squared of the new components is the relative intensity of spin-1 mixed into the original spin-0 state, it is equal to

$$\left(\frac{\Delta k^*}{m}\right)^2.$$

As mentioned in the introduction to this Sect. 5 the appearance of the spin neighbor produces a spin spread, not unlike the spreads in energy and momentum, which, however, have a continuous appearance, corresponding to the non-compact structure of the underlying group of spacetime translations. In order to make a comparison between the continuous and the discrete spreads we define a measure of the spin spread $\Delta S$ as the root-mean-square variation of the spin of the short- lived state. To first order in $\Gamma/m$ we find for the example at hand:

$$\Delta S = \frac{|\Delta k^*|}{m} \geq \frac{\Gamma}{2m} \tag{5.35}$$

Under these circumstances, different spin states no longer have to be orthogonal to each other. If one of them is short-lived and therefore has spin neighbors, we designate it $|j, m, \Gamma\rangle$ and write

$$\langle j_1, m_1 | j_2, m_2, \Gamma\rangle \neq 0 \text{ even if } j_1 \neq j_2 \text{ or } m_1 \neq m_2, \tag{5.36}$$

provided they have some overlap in the enlarged spin space that contains the spin spread of the short-lived particle.

## 6 Conclusions and outlook

In this paper we have shown how the two fourvectors of energy-momentum average and spread govern the behaviour of relativistic resonances.

Also we have used Wigner's method of Poincaré group representations that are induced by representations of the space-time translation and rotation groups, to explain the occurrence of neighbouring spins as an immediate consequence of a short lifetime of relativistic states.

Whereas it is clear that short lived states do not have to be orthogonal to each other and to the stable particles with the same charge-like quantum numbers – and the appearing spin spread makes this possible even for different spins – it is less clear at the moment how to interpret this situation in theoretical and in quantitative terms.

For the quantum probability interpretation of experiments one uses absolute squares of 'probability amplitudes' (transition elements), i.e. one relies on the scalar product in the Hilbert space for the states. This positive definite scalar product is a consequence of the $\mathbf{U}(1)$-group in which the time development of the states is represented, $t \longmapsto \mathrm{e}^{-iEt} \in \mathbf{U}(1)$. What happens for unstable states with a noncompact time development $\mathrm{e}^{-i(E - i\frac{\Gamma}{2})t} \notin \mathbf{U}(1)$ for $\Gamma > 0$? Is there no Hilbert space? How do we have to interpret experiments with unstable particles?

A stable and an unstable particle with a non-zero scalar product between them will show quantum mechanical interference so that their cross-section contains the amplitude not only as the magnitude squared but also linearly. The interference term produces decay products. This seems to be the case in quasidiffractive scattering[5]. For a correct description of the situation, all states that have a non-zero scalar product between them are partially identical and must be included in the counting. 'Probability collectives'[6] have to be formed.

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[5] Quasidiffractive scattering through partial identity has been presented by one of us (W.B.) at the Workshop on Resonances and Time Asymmetric Quantum Theory in Jaca [9] (Spain), 30 May to 4 June 2001

[6] Probability collectives have been presented by one of us (H.S.) in a talk on the same workshop [10]